Formulas for 1D Perfectly Elastic Collisions

Assume that masses m_1 and m_2 are moving along the x-axis at velocities v_1 and v_2, respectively. Then they collide. Conservation of momentum tells us:

$$m_1 v_1 + m_2 v_2 = m_1 \tilde{v}_1 + m_2 \tilde{v}_2$$

where v indicates a velocity before the collision, and $\tilde{v}$ indicates a velocity after the collision.

Since we are assuming a perfectly elastic collision, it must also be the case that the total kinetic energies before and after the collision are equal. We have:

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 \tilde{v}_1^2 + \tfrac{1}{2} m_2 \tilde{v}_2^2$$

We thus have two equations in two unknowns, and the rest is merely algebra. However, let's write out the "merely", since it is a bit involved. To save keystrokes, I will define $\mu = m_1/m_2$, and then I will divide through both equations by m_2. This yields:

momentum equation: $\mu v_1 + v_2 = \mu \tilde{v}_1 + \tilde{v}_2$
kinetic energy equation: $\mu v_1^2 + v_2^2 = \mu \tilde{v}_1^2 + \tilde{v}_2^2$

Regrouping both equations:

1) $\mu(v_1 - \tilde{v}_1) = \tilde{v}_2 - v_2$
2) $\mu(v_1^2 - \tilde{v}_1^2) = \tilde{v}_2^2 - v_2^2$

We now note that $x^2 - y^2 = (x + y)(x - y)$, so equation (2) becomes:

3) $\mu(v_1 + \tilde{v}_1)(v_1 - \tilde{v}_1) = (\tilde{v}_2 + v_2)(\tilde{v}_2 - v_2)$

We can divide both sides of equation (3) by the respective side of equation (1) to get:

4) $v_1 + \tilde{v}_1 = \tilde{v}_2 + v_2$, which is a pretty amazing equation, since it tells us that we have a sort-of conservation of velocity between the two masses.

We then solve equation (4) for $\tilde{v}_2$, and substitute back into equation (1):

5) $\mu(v_1 - \tilde{v}_1) = (v_1 + \tilde{v}_1 - v_2) - v_2$, or $\mu v_1 - \mu \tilde{v}_1 = v_1 + \tilde{v}_1 - 2v_2$

A little rearrangement yields $\tilde{v}_1 = [2v_2 + (\mu - 1)v_1] / (1 + \mu)$. Finally, multiplying top and bottom by m_2 gives us:

$$\tilde{v}_1 = [2m_2 v_2 + (m_1 - m_2)v_1] / (m_1 + m_2)$$

Since it is evident that the masses must be symmetric under mirror inversion, we can immediately write down an expression for the velocity of the second mass*:

$$\tilde{v}_2 = [2m_1 v_1 + (m_2 - m_1)v_2] / (m_1 + m_2)$$

*This is the sort of thing physicists say *after* they've worked all the algebra and see that the solution for the second velocity is the same as the first with all the subscripts (1,2) reversed.