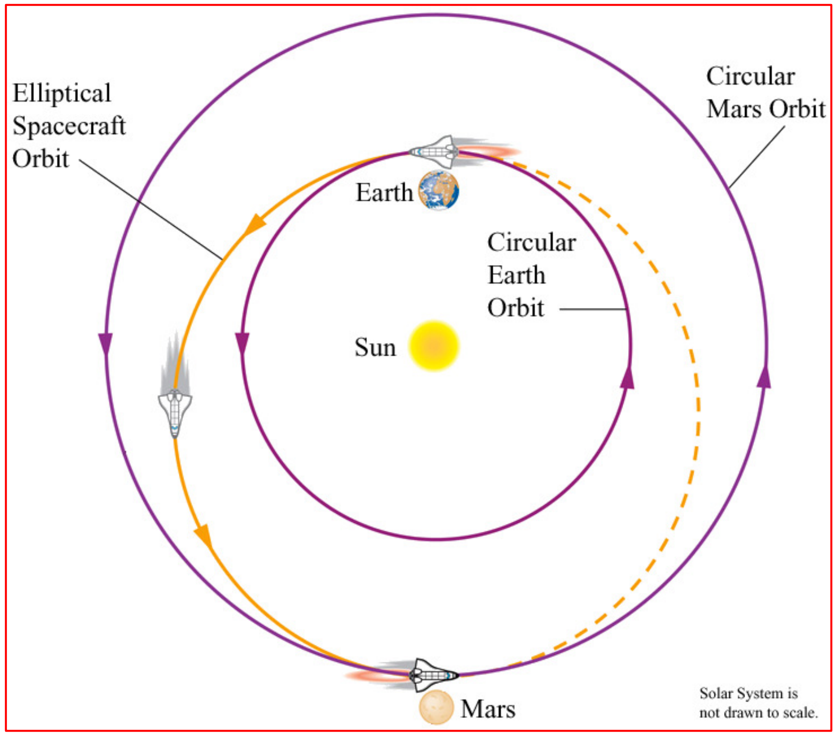
A Trip To Mars



It is the future. We have a spaceship that is scheduled to leave Earth and arrive on Mars just in time for the semi-annual Robert A. Heinlein Podkayne Of Mars Festival.

Unfortunately, it is not so far in the future that the spaceship can be taken to Mars using "thruster power", as though traveling to another planet is no different than driving down to the 7-11 for a Slurpee.

We must follow Sir Isaac's Laws, which are appallingly restrictive when it comes to vacationing on Mars. First one must leave planet Earth. This requires that you blast off with a *minimum* speed (escape velocity) of $v^2 = 2GM_E/r_E$, where:

$G = 6.674 \times 10^{-11}$ N m^2/kg^2

M_E = Earth's mass = 5.974×10^{24} kg

r_E = Earth's radius = 6.371×10^6 m

which gives us $v = 1.119 \times 10^4$ m/s, or 25,026 miles per hour. This is about ten times the speed of a high-velocity military rifle bullet, so you need about 100 times as much kinetic energy per kilogram to lift off the Earth than you do to place an armor-piercing round through a half-inch of steel at a distance of two miles. And then once you do reach high Earth orbit, your velocity relative to the Earth is almost nothing because most of the lift-off kinetic energy is converted into gravitational energy.

Meanwhile, the Earth is orbiting the Sun at a speed of 29.78 km/s, or 66,600 miles per hour, which is 2.66 times your escape velocity, which you no longer have. This means that for all practical purposes, after your spaceship reaches space (as beautifully drawn beside the tiny Earth in the figure), you will be moving at 66,660 mph tangent to the Earth's orbit, and at a completely insignificant smidgen of mph radially. [Now you know why NASA always sends their spacecraft on looping, spiral paths to other planets. The energy it would take to "kill" the momentum imparted by Earth's velocity and cut "straight across" the solar system is something like 266 times as much per kilogram as that in a (laughably so-called) high velocity Army bullet.]

We have no choice but to let our spaceship spend almost all of its time coasting from Earth to Mars. Here is the idea: we will place it into an elliptical orbit *around the Sun* such that its closest approach (to the Sun) coincides with the distance of Earth's orbit, but its farthest point (from the Sun) coincides with the distance of Mars' orbit. This is illustrated in the figure. These requirements place very rigid limits on our orbital parameters. If the ellipse is to reach from Earth orbit on the one side (Earth-Sun distance = R_E) to Mars orbit on the other side (Mars-Sun distance = R_M), then we must have $2a = R_E + R_M$, where a is the semi-major axis of the ellipse. End of story. That must be our semi-major axis.

Browsing through the **Orbital Stuff** handout (conveniently available on the class website), we find the useful little formula $v^2 = GM_S(2/r - 1/a)$, which gives the velocity at any radius r for an object in an elliptical orbit. In this case, since Earth must be at Point A in the illustration of an ellipse in the **Orbital Stuff** handout, we know that our spacecraft will begin its journey with $r = R_E$. Substituting for r and a gives: $v^2 = GM_S[(2/R_E - 2/(R_E + R_M)]$, and a bit of algebra yields $v^2 = (GM_S/R_E)[2\beta/(1+\beta)]$, where $\beta = R_M/R_E = 1.5237$ = the ratio of the orbital axes for Mars and Earth. This is the speed the spacecraft must have (at Point A on the ellipse) if it is to enter a Mars-Earth elliptical orbit. Using M_S = mass of Sun = 1.9891×10^{30} kg, and R_E = radius of Earth's orbit = 1.496×10^{11} m, we calculate $v = 32.72$ km/s, or 73,185 mph.

On the one hand, yikes. That is a lot of velocity. On the other hand, fortunately, the spacecraft is already moving at 66,600 mph as it sits in high Earth orbit, so we only need to *add* 73,185 mph – 66,600 mph = 6,585 mph to its speed. That isn't *so* bad, only about 26% of the speed needed to escape from Earth's surface in the first place.

So, we boost our spacecraft (on a line *tangent* to the Earth's orbit) to a velocity of 73,185 mph (relative to the Sun), and away we go. Now, how long do we have to wait? Well, Kepler's 3rd Law says $P^2 = (4\pi^2/GM_S)\ r^3$ for circular orbits around the Sun. If one does a very general analysis that takes into account all the ways that a particle can move in a $1/r^2$ force field, then you find that, for an ellipse, $P^2 = (4\pi^2/GM_S)\ a^3$.

Yep. That's right. Just exchange r for a and then you have Kepler's 3rd Law for an ellipse. Better yet, if you divide Kepler's 3rd by itself except using the appropriate numbers for Earth's orbit, you have $P^2 = a^3$, where P is in years instead of seconds, and a is in units of AU rather than in meters. (AU, for those who may be astrophysically challenged, is the abbreviation for Astronomical Unit. One AU = the radius of Earth's orbit.) Since for our ellipse we have $2a = R_E + R_M = 1 + 1.5237$ (in AU units), $a = 1.26185$ and $P = (1.26185)^{3/2} = 1.417$ year. This is the orbit's period, but we only need half a period to get to Mars, so the travel time is $1.417 / 2 = 0.709$ yrs = 8 ½ months.

And that, I am afraid, is that. Sir Isaac says it will take you 8 ½ months to get to Mars if you use this method, and he tends to be remarkably inflexible about these things. This kind of transfer orbit, in which the ellipse *just* touches the inner and outer circular orbits you are moving between, is known as a *Hohmann transfer orbit*. The "just touching" part means we have used an ellipse with the minimum possible semi-major axis a – but minimum possible axis also means minimum possible (most negative) mechanical energy. Therefore this orbit is more energy efficient than any other orbit, unless you know a way to abolish the Sun's gravitational potential. Want to reach Mars in less than 8 ½ months? It will cost you plenty in additional energy, I guarantee.

How fast will we be moving, when we reach Mars? One could re-calculate the velocity using the formulas on the previous page, but as a shortcut, let's invoke conservation of momentum. Because the Earth and Mars are at either end of the semi-major axis, we know $\mathbf{L} = \mathbf{r} \times \mathbf{p} = mvr \sin\theta$ will just be mvr, because $\sin\theta = 1$ at the ends. We have $mv_E R_E = mv_M R_M$, or $v_M = v_E R_E / R_M = 73{,}185\ \text{mph}/1.5237 = 48{,}031\ \text{mph}$. This is an amusing result, because Mars' orbital speed is 53,980 mph. This means the spacecraft needs to intersect the Martian orbit *in front* of Mars, and then let Mars catch up with it! Also, it means that when the spacecraft eases into position, Mars will be thundering towards it (hope the Captain remembers to check the rear-view mirror) at a speed of 53,980 – 48,031 = 5,949 mph. To land on Mars, the spacecraft must negate this velocity differential in some way: blazing like a meteor into the thin Martian atmosphere and using some form of air braking is a popular option. It would be much more expensive to soft-land on Mars if it didn't have an atmosphere, because then you would have to drag fuel tanks all the way out there and boost your speed by 5,949 mph to match that of the red planet.