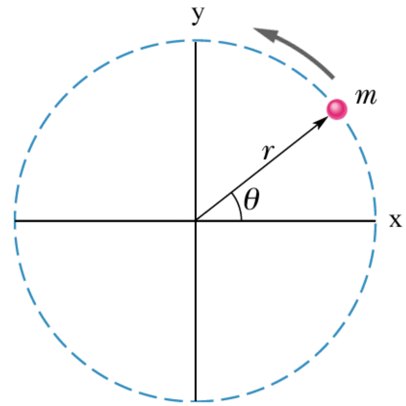
Constant Circular Motion

Let us prove that an object moving in a circle at constant speed has an acceleration directed towards the center of the circle. The easy way to do this is by using the basic definitions of velocity and acceleration, and a little calculus. Consider the particle *m* shown below. Its position is given by the vector $\boldsymbol{r} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$. (I've put vectors in bright red to make them easier to read.) If the particle is rotating at constant speed, then θ must be changing at a constant rate. This means $\theta = \omega t$ where ω is a constant that we call the *angular velocity*, in analogy with $x = v_o t$. The units for ω are radians/sec. We can convert ω into more familiar "cycles per second" (aka, the frequency) by noting that one time around a circle represents a rotation of 2π. So $\omega = 2\pi f$.



The moving particle is therefore described by:

$\boldsymbol{r} = r\cos(\omega t)\,\hat{\imath} + r\sin(\omega t)\,\hat{\jmath}$

Using the basic definition of velocity ($\boldsymbol{v} = d\boldsymbol{r}/dt$) gives:

$\boldsymbol{v} = -\omega r\sin(\omega t)\,\hat{\imath} + \omega r\cos(\omega t)\,\hat{\jmath}$

Using the basic definition of acceleration ($\boldsymbol{a} = d\boldsymbol{v}/dt$) gives:

$\boldsymbol{a} = -\omega^2 r\cos(\omega t)\,\hat{\imath} - \omega^2 r\sin(\omega t)\,\hat{\jmath} = -\omega^2\,[r\cos(\omega t)\,\hat{\imath} + r\sin(\omega t)\,\hat{\jmath}] = -\omega^2\,\boldsymbol{r}$.

In other words, the $\boldsymbol{a}$ vector for particle *m* is exactly the same as the $\boldsymbol{r}$ vector, except pointed in the opposite direction and multiplied by ω^2. Since the $\boldsymbol{r}$ vector always points outward from the center, the $\boldsymbol{a}$ vector must always point inward! Note that this derivation makes no assumptions about *why* the particle is moving in a circle; it relies only on the basic definitions of motion.

Also, this result shows us that the magnitude of the $\boldsymbol{a}$ vector is $a = \omega^2 r$. Since the speed of a particle moving in a circle = (distance around the circle) X (times per second the particle circles the circle), we have $v = (2\pi r)$ X frequency $= 2\pi r f = \omega r$, so $a = \omega^2 r = (v/r)^2 r = v^2 / r$.

This quantity is the infamous "centrifugal acceleration". If you are an ant riding on the particle, then in certain ways there seems to a force that is trying to push you radially away from the center of the circle. This is that "force". However, it is important to realize that the apparent outward "pull" of "centrifugal force" when you are riding on a roller coaster, or in a car going around a curve, etc, is largely an illusion. You in fact move tangentially to the radius if you fly off a rotating object, and it is only the *relative* motion between this tangential velocity and the point on the rotating object where you used to be which generates the illusion of an outward acting force.