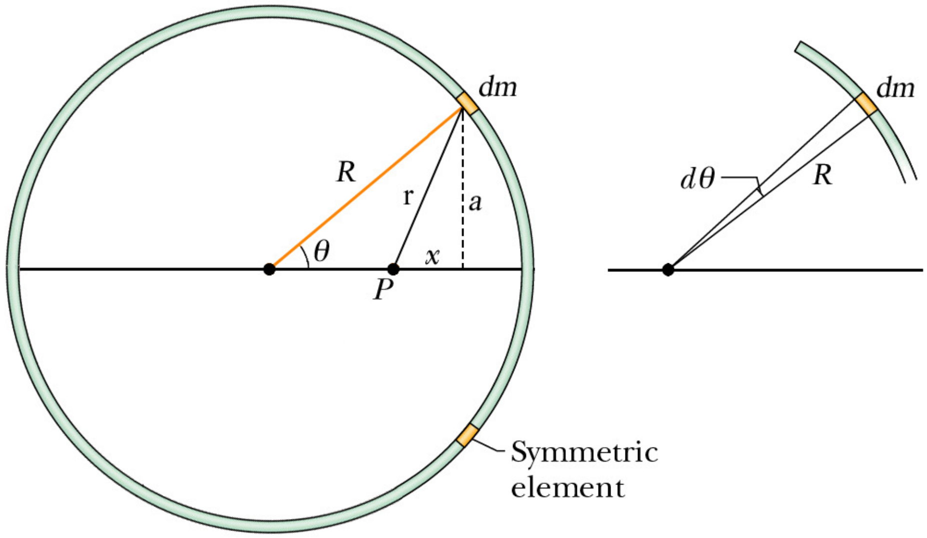
Gravity Inside A Spherical Shell



We wish to calculate the gravity at an arbitrary point P inside a thin, uniform, spherical shell of mass M. The figure at right shows the geometry we will use. We have spherical symmetry, so we can always draw a line through P and the center of the sphere. We will call this line the x-axis, and we will put the origin of the x-axis at P.

We start with $F = GMm / r^2$.
We can assume a small mass m_0 to be at point P, but the mass in the sphere is at varying distance from P, so we must use $dF = Gm_0\, dm / r^2$ and come up with a way to integrate over all the infinitesimal dm's.

The yellow element labeled dm in the figure, and the one below it labeled "symmetric element", are parts of a circle of radius a that runs around the sphere. We will choose this circular slice to be our dm for one very good reason: all parts of the slice are at an equal distance r from point P. Note that for any element on the circle there is an identical element on the opposite side of the circle (as illustrated by the two yellow spots) so the force components *perpendicular* to the x-axis always cancel. The net gravitational force at P is only along the x-axis. This reduces our apparently 3D problem to a 1D integral where $dF = Gm_0\, dm / r^2$ multiplied by the cosine of the angle between r and the x-axis, which is just x/r. We have $dF = Gm_0\, x\, dm / r^3$ as the force at P.

We want to integrate along x, so let's eliminate r from the equation for dF. We note from the figure that $r^2 = x^2 + a^2$ and $\mathbf{R^2 = (P + x)^2 + a^2}$. Solving the red equation for a^2 and substituting it into the green equation gives us $r^2 = x^2 + R^2 - (P + x)^2 = x^2 + R^2 - P^2 - x^2 - 2Px = R^2 - P^2 - 2Px$. We thus arrive at $dF = Gm_0\, x\, dm / (R^2 - P^2 - 2Px)^{3/2}$ as the force at P.

Turning our attention to dm, we know from similar problems with uniform objects that dm can be written as a ratio: dm / M = (area of circular slice)/(area of the sphere), or $dm = (M\, dA)/(4\pi R^2)$. To find dA, we note that the radius of the slice is a, so its length is $2\pi a$. The width of the slice (as shown in the small inset) is $Rd\theta$, so $dA = 2\pi a\, Rd\theta$. From the figure we have $R \sin\theta = a$, so $dA = 2\pi R^2 \sin\theta\, d\theta$.

We now have $dm = M(2\pi R^2 \sin\theta\, d\theta)/(4\pi R^2) = ½\, M \sin\theta\, d\theta$. Well and good, but we need to integrate over dx, not $d\theta$. Fortunately, the figure shows us that $R \cos\theta = P + x$, so taking the derivative of both sides yields $-R \sin\theta\, d\theta = dx$. Substitution for $d\theta$ in the equation for dm gives us $dm = -½\, (M/R)\, dx$.

Our equation for the differential force now reads $dF = -(GMm_0/2R)(x\, dx)/(R^2 - P^2 - 2Px)^{3/2}$. We finally have nothing in the equation except x and constants, so we are ready to integrate. We look up the integral in our favorite table and find $F = -(GMm_0/2RP^2)(R^2 - P^2 - Px)/(R^2 - P^2 - 2Px)^{1/2}$.

We set $x = 0$ to be at P, so our limits of integration run from $x = R - P$ to $x = -R - P$. Evaluating the upper limit: $(R^2 - P^2 + PR + P^2) / (R^2 - P^2 + 2PR + 2P^2)^{1/2} = R(R + P) / [(R + P)^2]^{1/2} = R$. Evaluating the lower limit: $(R^2 - P^2 - PR + P^2) / (R^2 - P^2 - 2PR + 2P^2)^{1/2} = R(R - P) / [(R - P)^2]^{1/2} = R$. We are left with $F = -(GMm_0/2RP^2)(R - R)$ = zero!!

Since our choice of point P was arbitrary, this result holds for anywhere inside the shell, whether you are one millimeter from the wall or in the center. And since a thick shell can be thought of as a series of nested thin shells, the result still holds regardless of how thick the shell is. We don't even need the thin shells to be made of the same material: the total mass M is outside the integral and has nothing to do with the cancellation of forces, so it matters not if the sphere is made of paper or lead. The only thing that matters is that the shells be spherically symmetric.

But, beware when applying this result to a solid sphere: if you descend into the Earth (for example), then only that part of the Earth which is *above* you forms a spherical shell where the net gravitation cancels to zero. The part of the Earth still below you will still exert an attractive force with whatever mass it has. The only spot inside a solid sphere where the net gravity is zero is the center.

For a uniform sphere (which the Earth isn't), it is easy to derive how the gravity must vary as one moves down into the sphere. The mass fraction of a uniform sphere which is inside a radius r is just the ratio of the volume inside r compared to the volume of the entire sphere, or $m / M = (^4/_3\pi r^3) / (^4/_3\pi R^3) = (r/R)^3$. We have just finished proving that all the mass outside r must produce zero gravity, so the gravity is entirely due to the mass inside r, or $F = Gm_1m_2/r^2 = Gm_1[M\ (r/R)^3] / r^2 = Gm_1Mr / R^3$. We recognize that GM / R^2 is the (constant) surface gravity of the sphere, so $F/m_1 = g\ (r/R)$. The acceleration due to gravity for a uniform sphere is g at the surface, and decreases linearly to zero at the center.