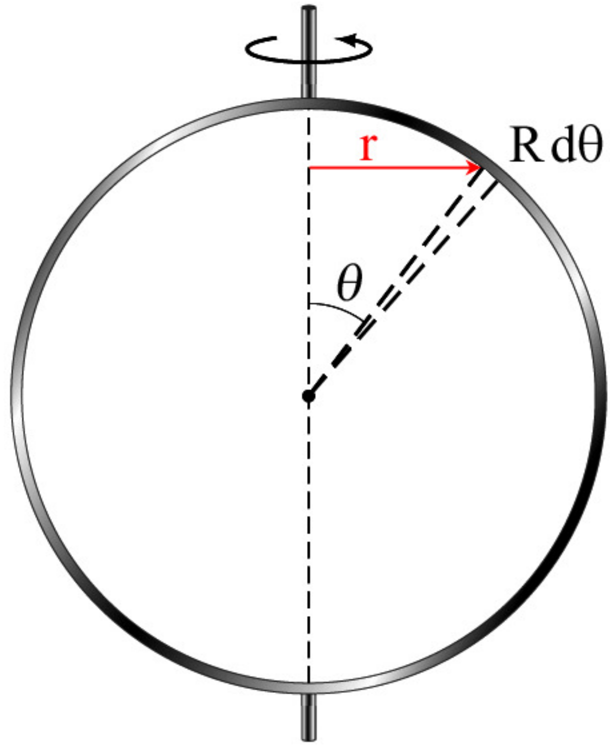
Moment Of Inertia Of A Hollow Sphere



We will calculate the moment of inertia of a thin, hollow, spherical shell of mass M and radius R, with an axis of rotation through the center of the sphere. We start with the definition of moment of inertia for an infinitesimal mass: $dI = dm\, r^2$. We will need to set up an integral to find I.

The r in the expression for dI is the distance from the axis of rotation to any element on the sphere's surface. This is shown in red on the diagram.

dm is an unknown. However, since the sphere is uniform, we deduce that dm and the total mass M have the same ratio as the infinitesimal area dA (occupied by dm) and the total surface area of the sphere. In symbols: $dm / M = dA / 4\pi R^2$.

Now we need to calculate dA. From the diagram we see that the element dm, which is defined as being a distance r from the axis, is in fact a ring in this geometry, not a point. So, $dA = 2\pi r\, ds$, where ds is some infinitesimal width. If the ring were on a flat disk, it would be fairly obvious that $ds = dr$. But, it isn't. Areas on the outside of a sphere are measured by how far down the sphere they extend, not by how far they are from the axis. In our case we need a distance moving away from the pole, or $s = R\theta$. Then $ds = R\, d\theta$ as shown.

Putting everything together, we have $dI = (M / 4\pi R^2)(2\pi r\, R\, d\theta)\, r^2$. Mixing r and θ in one integral will not do, so we note that $r = R \sin\theta$ and substitute to get $dI = (M / 4\pi R^2)(2\pi R \sin\theta\, R\, d\theta)\, R^2 \sin^2\theta = \frac{1}{2} MR^2 \sin^3\theta\, d\theta.$

Now we need only integrate to find I. $\sin^3\theta$ is a tedious function to integrate by hand, so we look in our favorite table and find that the integral is $\frac{1}{12} \cos(3\theta) - \frac{3}{4} \cos\theta$. As the diagram shows, θ must run from 0 to π if we integrate over the entire sphere. The final bit of arithmetic is:

$$I = \tfrac{1}{8} MR^2\{[\tfrac{1}{3} \cos(3\pi) - 3\cos(\pi)] - [\tfrac{1}{3} \cos(0) - 3\cos(0)]\} = \tfrac{1}{8} MR^2\{(-\tfrac{1}{3} + 3) - (\tfrac{1}{3} - 3)\} = \tfrac{2}{3} MR^2.$$

With regard to finding dA, one might wonder why you cannot use the formula for the surface area of a sphere, $A = 4\pi r^2$, which gives $dA = 8\pi r\, dr$. The answer is that this differential tells you the rate at which the entire surface area of the sphere is changing if you happen to be inflating it by a factor of dr. Very interesting, no doubt, but not what we needed. The surface area of the portion of a sphere subtended by any angle θ is given by the quite obscure function $A = 2\pi R^2(1 - \cos\theta)$. As you can readily verify, the derivative of this function does indeed give $dA = 2\pi R^2 \sin\theta\, d\theta$, in agreement with the derivation above.