Orbital Stuff

Note – I will use M_E and R_E for the mass and radius of the Earth to avoid confusion, but anywhere you see these, the mass and radius for any other planet or star can be substituted.

Gravitational potential energy for an object near planet Earth: $U = -GmM_E / r$, where:
$G = 6.674 \times 10^{-11}$ N m^2/kg^2, m = mass of the object, M_E = mass of Earth, r = distance from the object to the center of the Earth.

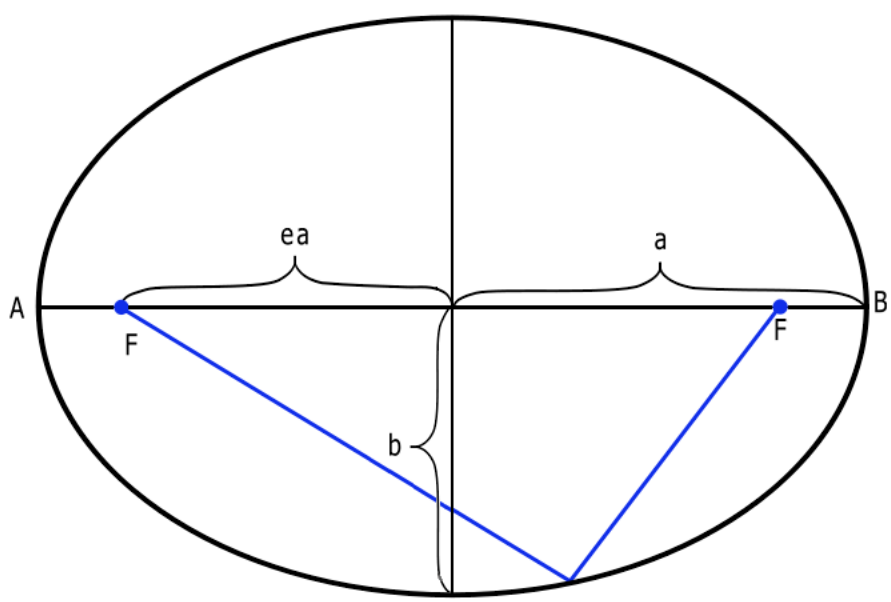
Escape speed from the surface of the Earth:
Total mechanical energy must equal zero: $0 = -(GmM_E / R_E) + \frac{1}{2} mv^2$, or $v = (2GM_E / R_E)^{1/2}$

Condition for a stable circular orbit:
Gravity equals centrifugal force: $GmM_E / r^2 = mv^2 / r$, or $v = (GM_E / r)^{1/2}$
This condition means that for a satellite in orbit at a radius r, it can *only* have a speed of exactly v.

Kepler's Third Law
If we substitute $v = \omega r$, $\omega = 2\pi f$, and $P = 1/f$ into the equation for a circular orbit, then we obtain: $v = \omega r = 2\pi fr = 2\pi r/P = (GM_E / r)^{1/2}$, or $P^2 = (4\pi^2/GM_E)\, r^3$, which is the traditional way of writing down this law. The tradition comes about because Kepler originally wrote the Law as a ratio for the planets: $P_1^2 / P_2^2 = r_1^3 / r_2^3$, where the subscripts 1,2 refer to different planets. In other words, Kepler was using $(4\pi^2/GM_S)$ for his constant, where M_S is the mass of the Sun, but he did not know it. By using a ratio he had cancelled out the constant – which he had to do, since he had no possible way of measuring it.

Total mechanical energy of a circular orbit:
$T = U + E_K = -(GmM_E / r) + \frac{1}{2} mv^2$. But for a circular orbit $v = (GM_E / r)^{1/2}$, so substituting:
$T = -(GmM_E / r) + \frac{1}{2} m(GM_E / r) = - GmM_E / 2r$

Kepler's First Law
The planets move in elliptical orbits with the Sun at one focus. In case it has been awhile since you studied the particulars of the ellipse, it is defined as a curve such that the sum of the distances r_1 and r_2 from two points (called the foci) to the curve is a constant: $r_1 + r_2 = 2a$. The long axis "a" is called the semi-major axis, the shorter perpendicular axis "b" is called the semi-minor axis. The traditional way of drawing an ellipse is to loop a string around two pins stuck in a sheet of paper, then keep the string drawn tight as you trace out a curve. The eccentricity of the ellipse is defined as how far the foci are from the center, or $f = ea$. $e = 0$ for a circle, and $e = 1$ for an ellipse so thin that it is basically a straight line. A bit of trigonometry using the geometry shown at right gives $e^2 = 1 - b^2/a^2$.



Total mechanical energy of an elliptical orbit:

Believe it or not, $T = -GmM_E / 2a$, where a is the semi-major axis. This is the same formula that we wrote down for a circular orbit, except that a has replaced r !!! The mathematical machinery necessary to rigorously derive this result is better left for Phyx 330-1 (Mechanics), but the result itself is simple enough. Note that the eccentricity e of the ellipse does not appear in the formula: incredibly, it doesn't matter! The ellipse can be as fat or as squashed as you wish, only the length of "a" affects the total energy.

Kepler's Second Law

In its original form, Kepler's Second Law stated that if you draw a line from the Sun to any planet, and let the line "sweep out" a triangular area over a certain time, then that *same* area will be swept out by the planet (over the same amount of time) no matter where in its orbit the planet is.

My mind boggles at the thought of how much fiddling around with untold tens of thousands of hand calculations Kepler had to make to come up with this one. Fortunately, we do not need to duplicate his effort. As I showed in class, and as the textbook also proves, Kepler's Second Law is really only an amusing geometric consequence of the conservation of angular momentum. The critical point is that the speed of an orbiting object must vary if it moves along an elliptical path: the closer it is to the focus, the faster it moves.

The speed of an object in an elliptical orbit

We still have $T = U + E_K$, as we did for a circular orbit. However, now $T = -GmM_E / 2a$, which is a fixed quantity since a is fixed. Also, we *cannot* make the rather trivial assumption that $v = (GM_E / r)^{1/2}$, as we did for a circular orbit. Instead we have: $T = -GmM_E / 2a = -GmM_E / r + \frac{1}{2} mv^2$, and a bit of algebra gives us $v^2 = GM_E (2/r - 1/a)$. The speed of the object varies as it goes around the ellipse, as expected. Note that the variable r is the distance from the satellite to the focus point the Sun or Earth occupies.

Angular momentum of an elliptical orbit

The mechanical energy T of a satellite is independent of whether its orbit is fat and round, or squashed so flat it looks like a pancake. Thus, common sense says that some other physical quantity must be the determining factor for the eccentricity of the orbit. So without further ado, allow me to announce that this other quantity is (Ta-Da!): angular momentum.

$\mathbf{L} = \mathbf{r} \times \mathbf{p} = r\, mv \sin\theta$, so the flatter the ellipse then the shallower the angle between v and r, on average, and the less angular momentum you have. Circular orbits ($e = 0$) have maximum L, because $\sin\theta$ is always 1, and a bucket falling to Earth in a straight line ($e = 1$) has zero L, because the radius and the force are in the same direction, so $\sin\theta = 0$.

We can connect L to other orbit variables by looking at a special point on the ellipse. At Point "A" in the diagram above, the velocity vector of the satellite is oriented at exactly 90° to the radius. Hence $\sin\theta = 1$. Also, at that point, we can see that the distance between the satellite and the Earth is $r_A = a - ea$. Let's do a little algebra.

First insert r_A into the formula for the velocity in an elliptical orbit (red formula just above): $v^2 = GM_E [2/(a - ea) - (1/a)] = GM_E [(2a - a + ea)/(a - ea)a] = (GM_E/a)(1 + e)/(1 - e)$. Now we know the exact speed of the satellite at r_A.

Next, write $L = r\, mv \sin\theta = mvr_A = mv(a - ea)$. Since angular momentum is conserved, we know this value for L at r_A will be the value for L everywhere, so let's call it L_0. If we now substitute the formula for v at r_A: $L_0 = m(GM_E/a)^{1/2} (1 + e)^{1/2} (1 - e)^{-1/2} a(1 - e)$, where the green part is v. Canceling a factor of $(1 - e)^{-1/2}$, we continue to grind forward on the algebra: $L_0 = m(GM_E/a)^{1/2} (1 + e)^{1/2} (1 - e)^{1/2} a = ma(GM_E/a)^{1/2} [(1 + e)(1 - e)]^{1/2} = ma(GM_E/a)^{1/2} (1 - e^2)^{1/2} = ma(GM_E/a)^{1/2} (1 - 1 + b^2/a^2)^{1/2} = m(GM_E/a)^{1/2} b$, where I used $e^2 = 1 - b^2/a^2$ in that next to last step.

We can make this formula even simpler and write $L_0 = mv_0b$, where v_0 is the speed that the satellite would have if it were going in a circle of radius a. Note that L_0 is a maximum when $b = a$ (a circle), and zero when $b = 0$ (ellipse is flattened into a line).