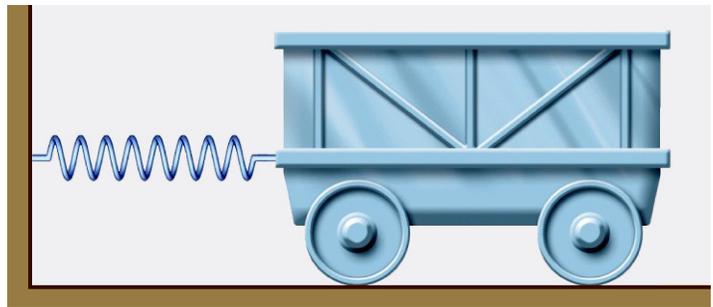


## Oscillation Frequency Of A Wagon

The basic theory behind oscillations comes from the force equation for a point mass  $m$  attached to a spring with constant  $k$ . This is just  $ma = -kx$ . If we write this equation out with a few more of the parentheses inserted, we have  $m[d^2x(t)/dt^2] = -kx(t)$ , where I have specifically written  $x(t)$  as a function of time. This is a differential equation, and you can deduce a solution to it by asking yourself what function, if you differentiate it twice, will come back to itself with a minus sign in front. One possibility is the sin function, so let us assume that  $x(t) = A \sin(\omega t)$ , where  $A$  is the amplitude of the sin function and  $\omega$  is its angular frequency. At first glance this solution makes sense, because it just has the point mass oscillating at constant frequency, and from observation we know this is what masses attached to springs do. But to be sure, let's substitute  $A \sin(\omega t)$  back into our force equation and see what we get:  $m[d^2x(t)/dt^2] = -kx(t)$  becomes  $m[d^2(A \sin(\omega t))/dt^2] = -kA \sin(\omega t)$ . The constant  $A$  then cancels out. For the left-hand side we take one time derivative of  $\sin(\omega t) = \omega \cos(\omega t)$ , and a second derivative gives  $-\omega^2 \sin(\omega t)$ . This leaves  $m[-\omega^2 \sin(\omega t)] = -k \sin(\omega t)$ , or  $\omega = (k/m)^{1/2}$ . In other words,  $x(t) = A \sin(\omega t)$  is indeed a solution to the force equation, IF  $\omega = (k/m)^{1/2}$ .

Since the solution to this differential equation is a matter of mathematics, not physics, we can generalize it to say that if one has *any* physical situation which is described by an equation of the form  $d^2\Psi(t)/dt^2 = -C \Psi(t)$ , where  $C$  is a positive constant and  $\Psi(t)$  is whatever variable you happen to be using, then the system is a harmonic oscillator with an angular frequency of  $\omega = C^{1/2}$ .

That brings us to our wagon. Problem 15-31 in the textbook asks you to calculate the oscillation frequency for wagon attached to a spring. We have the following parameters: wagon mass =  $m$ , mass of each wheel =  $M$ , radius of each wheel =  $R$ , spring constant =  $k$ . There is no friction, except for static friction so that the wheels do not slip as they roll.



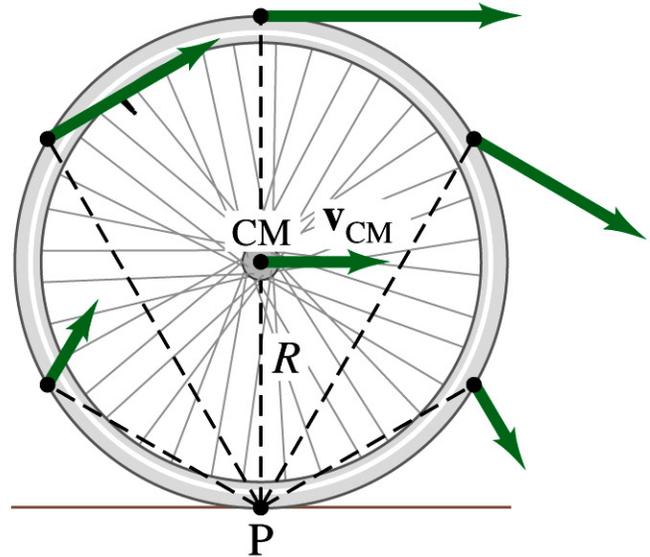
The strategy for finding the oscillation frequency of a one-dimensional system is to work towards deriving an equation of the form  $d^2\Psi(t)/dt^2 = -C \Psi(t)$ , because then  $\omega$  can be immediately set equal to  $C^{1/2}$ . There are at least three ways of tackling Prob 15-31. One way is the solution given by the textbook's authors, but that one invokes virtual frictional forces which only exist as long as nothing is moving, and it also requires clairvoyance to get the signs right. I'm not going to bother with it.

The second way is simple algebraically, but requires very careful thought to see how it works. We start by noting that for any force you put on it, the body of the wagon will accelerate by  $a = F/m$ . Simple enough. And if I were to dump four wheels, each of mass  $M$ , into the cart, then it would accelerate by  $a = F / (m + 4M)$ . Also simple. But what if – the wheels are *rolling* rather than just rattling along for the ride? That means rotational inertia is present, and we must take it into account. For a wheel that is rolling without slipping, one has a complicated situation where the wheel is pivoting around a single point at any given instant, but also *shifting* the position of the pivot at every



instant! This definitely calls for some good illustrations. The image at left shows a stop-motion photo of a rolling bicycle wheel, and as you can see, the bottom of the wheel seems to be standing still, whereas the top is a fast-moving blur. This is what you would expect. The point on the wheel in contact with the road is instantaneously at rest, because the forward velocity of the wheel ( $v_{CM}$ ) and the tangential velocity of the point on the road ( $v_T = R\omega$ ) are equal and opposite. The excellent drawing just below (taken from a textbook by Douglas Giancoli) shows

the net velocity vectors for a rolling wheel, i.e., the green vectors include both the forward motion of the wheel ( $v_{CM}$ ) and the rotation of the wheel. The maximum speed of a point on the rim is achieved at the top, where the rotation and  $v_{CM}$  are in the same direction. The minimum speed of zero is achieved at the bottom where the rotation and  $v_{CM}$  are opposed.



In other words, a rolling wheel rotates around the point **P** as illustrated at right, *not* around its center. Also, and DON'T READ ON UNTIL YOU HAVE THOUGHT ABOUT THIS, the movement around the pivot incorporates *both* the linear and rotational motions of the wheel. (I repeat, both motions are included in the following analysis.)

To explore how this monster of vector addition reacts to an applied force, we must use the rotational equivalent of  $F = ma$ , i.e.,  $\tau = I\alpha$ .

We have  $\tau = \mathbf{r} \times \mathbf{F} = RF$ , where  $\mathbf{F}$  is applied at the axle and  $\mathbf{R}$  runs from point **P** up to the axle. ( $\mathbf{F}$  is at the axle because the axle is where the wheel is attached to the wagon.) The moment of inertia  $\mathbf{I}$  is a bit tricky, because the pivot point is on the ground, at point **P**. However, we can use the parallel axis theorem to calculate  $\mathbf{I}$  for a disk rotating around its edge:  $I = I_{CM} + Md^2 = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2$ .

Remembering that  $\mathbf{a} = R\alpha$ , we therefore see that  $\mathbf{r} \times \mathbf{F} = I\alpha$  reduces to  $RF = \frac{3}{2} MR^2(a/R)$  in this case, or  $F = \frac{3}{2} Ma$ . Let us be crystal clear about what we have just derived. If I place a linear force  $\mathbf{F}$  at the axle of a *sliding* wheel of mass  $m$ , it will accelerate by  $\mathbf{a} = F/m$ . But if I place that same force on the axle of a *rolling* disk of mass  $m$ , it will accelerate by  $\mathbf{a} = F/(\frac{3}{2} m)$ . The addition of rotation means that I have, in effect, “enhanced” the wheel’s inertia by a factor of  $\frac{3}{2}$ . A rotating wheel of 1 kg acts as though it has the inertia of a sliding wheel of 1.50 kg. This effective change in inertia is exactly why objects of different shapes (ball, disk, hoop) roll down an inclined ramp at different speeds.

Returning to our wagon, we note that the wagon’s body and its wheels are bolted together, so they must experience the same linear acceleration. But since the four rotating wheels act as though they have an enhanced inertia of  $\frac{3}{2}$  (as compared to a similar, non-rotating mass), we cannot just add their masses to the mass of the wagon and say that  $\mathbf{a} = F / (m + 4M)$ . No. Instead, we must accommodate their increased inertia and write  $\mathbf{a} = F / [m + (\frac{3}{2})(4M)] = F / (m + 6M)$ . Then, using  $F = -kx$  for the spring, we have:  $\mathbf{a} = d^2x/dt^2 = -kx/(m + 6M)$ , which means  $\omega^2 = k/(m + 6M)$ .

If the logic behind the above analysis seems a bit elusive, there is another way to solve the problem that requires calculus, but is more straight-forward. We start by writing down the total energy of the system. This includes the potential energy of the spring, and the kinetic energy of the wagon and its wheels:  $\frac{1}{2} kx^2 + \frac{1}{2} mv^2 + 4(\frac{1}{2} Mv^2) + 4(\frac{1}{2} I\omega^2) = E_0$ , where  $\mathbf{v}$  is the linear velocity of both the wagon’s body and wheels,  $m$  is the mass of the body,  $M$  is the mass of each wheel,  $\mathbf{I}$  and  $\omega$  are for a rotating disk, and  $E_0$  is the (unknown) total energy of the system. For the wheels, we have used the neat theorem that the kinetic energy of a moving, rotating body can always be separated into a linear part and a rotational part about the center of mass.

We know  $I = \frac{1}{2} MR^2$  and  $\mathbf{v} = R\omega$  for a disk. Substituting these into the energy equation gives us:  $\frac{1}{2} kx^2 + \frac{1}{2} mv^2 + 2Mv^2 + 2(\frac{1}{2} MR^2)(v/R)^2 = E_0$ , or  $\frac{1}{2} kx^2 + (\frac{1}{2} m + 3M)v^2 = E_0$ .

Let us now take the time derivative of the energy. Remembering the chain rule from calculus:  
 $(2)(\frac{1}{2} kx)(dx/dt) + (2)(\frac{1}{2} m + 3M)v(dv/dt) = 0$ . But  $dx/dt = v$ , so cancelling through leaves  
 $kx + (m + 6M)(dv/dt) = 0$ , or  $dv/dt = d^2x/dt^2 = -kx/(m + 6M)$ , and  $\omega^2 = k/(m + 6M)$ , as before.

Most physics textbooks do not mention this little trick of taking the time derivative of the energy to find oscillation frequencies, probably because it is somewhat limited in where it can be used. For the true mathematics fans among you, here are the caveats:

- 1) The problem must be one-dimensional. That does not mean it has to be linear, because a single rotation angle is equivalent to one dimension. It means that the position of your oscillating mass must be confined in such a way that only one free coordinate is needed to specify it.
- 2) Mechanical energy must be conserved. No friction, no dissipation.
- 3) All forces applied to the mass(s) must depend upon *space* only, not time. This means that  $F = -kx$  or  $F = Az^{3/2}$  are OK.  $F = F_0 \tan(\omega t)$  or  $F = kx + ct^4$  are not OK.