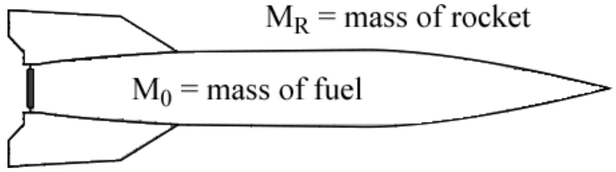
The Simple Rocket Equation II



Let's derive the rocket equation again, this time using conservation of momentum rather than Newton's Laws. We again have a rocket of mass M_R filled with a mass M_0 of fuel. The exhaust velocity of the fuel relative to the rocket is v_0.

In the previous derivation, I needed to insert several simplifying assumptions (constant rate of fuel burning, all the fuel is burnt in one session) to make the math manageable. Here, all we need do is note that a small amount of fuel of mass dM exhausted at a speed of v_0 will carry a small amount of momentum $p = dM\, v_0$. The rest of the rocket, whose mass M is unknown because we don't know how much fuel has been burnt, will gain a small, equal, and opposite momentum $p = M\, dv$. The initial momentum of the system is zero, so conservation of momentum gives us: $0 = dM\, v_0 + M\, dv$. A bit of algebra yields: $-dv = v_0\, (dM/M)$, or $-v = v_0 \ln(M) + C$.

We evaluate the constant of integration by noting that the initial mass of the rocket plus fuel is $M_R + M_0$, while its initial velocity is 0. This gives us $C = -v_0 \ln(M_R + M_0)$. Substituting for C and doing a bit of algebra, we have: $v = -v_0 \ln(M) + v_0 \ln(M_R + M_0) = v_0 \ln[(M_R + M_0)/M]$.

Note that this result gives us the velocity of the rocket at any point in time as a function of its current total mass, regardless of the rate or manner in which its fuel has been expended! If we want to evaluate the formula for the final velocity, we just note that $M = M_R$ when all the fuel is gone, so

$$v_F = v_0 \ln[(M_R + M_0)/M_R] = v_0 \ln(1 + M_0/M_R).$$

This is exactly the same formula that we derived previously. However, this method of deriving it is much more elegant.