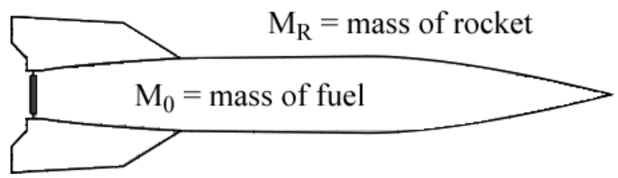
The Simple Rocket Equation



Consider a simple rocket – one which consists of a mass of fuel, a shell to hold the fuel, and not much else. (And that's a pretty good description, by the way, for both very large rockets like the Saturn V, and for very small rockets like the three-inch kind sold as home fireworks. Sheer speed is almost the only criteria at these extremes, so the rockets are the Newtonian equivalents of drag-racers: long on propellant and very short on turning ability.)

Here is our simple question: if I take the rocket to a weightless environment and let it burn up all its fuel, how fast will it be going?

We will start by letting M_R = the mass of the rocket and M_0 = the mass of the fuel. We can assume that the fuel burns at a constant rate of $dM/dt = R$, and exits the rocket with a constant velocity of v_0. It is then obvious that the force exerted by the burning fuel is $F = Rv_0$.

Well, OK. That last point may not be so obvious. Newton's first law is usually written as $\mathbf{F} = m\mathbf{a} = m(\Delta\mathbf{v}/\Delta t)$, but this must be a simplification because it assumes that the mass cannot change. As we discussed in class, the most general expression for force puts the mass *inside* the time derivative with the velocity, like so: $\mathbf{F} = d(m\mathbf{v})/dt = m(d\mathbf{v}/dt) + (dm/dt)\mathbf{v}$, where I've used the usual chain rule of differentiation. The first term in the chain is our old friend $m\mathbf{a}$, and the second term is none other than Rv_0, if you compare it to the constants we defined for our rocket fuel. In most situations dm/dt is zero and the $m\mathbf{a}$ term is all that counts, but for rockets the roles are reversed and the second term is the one we need because the mass is leaving the rocket but not accelerating.

The term $(dM/dt)v = Rv$ is called *thrust* by physicists. Don't let the special word fool you – thrust is just a force. It isn't hard to list any number of common examples: household fans generate thrust, as do propellers on ships and airplanes, jet engines, the air coming out of a punctured balloon, even the water spray in a kitchen sink or shower.

Returning to our physics problem, we can now write down an equation for the acceleration of our rocket. We have $\mathbf{F} = m\mathbf{a}$, or in this case: thrust = [rocket mass + fuel mass]($d\mathbf{v}/dt$). I have converted the $\mathbf{a}$ into a $d\mathbf{v}/dt$ because we want to know the rocket's final velocity, not its acceleration.

We know the thrust = Rv_0, so that leaves us with the rocket's mass. It starts off with a total mass of $M_R + M_0$, but this dwindles as the fuel is burnt. Since the rate of burning is constant, however, we know that $R \times t$ of the fuel will have been burnt after 't' seconds. Thus, the mass of the fuel left on the rocket will be $M_0 - Rt$. (Note that this derivation only works so long as there is fuel left. You can't burn negative fuel. When $M_0 - Rt = 0$, or $t = M_0/R$, the rocket's engine has to shut off.)

Thus our (differential!) equation for the velocity becomes $Rv_0 = [M_R + M_0 - Rt]\,(d\mathbf{v}/dt)$. To get from this to the final velocity we only have to collect terms to put dv on one side of the equation and dt on the other, then integrate:

$$v_F = \int_0^{\frac{M_0}{R}} \frac{Rv_0\,dt}{M_R + M_0 - Rt}$$

Note that the time on the definite integral cuts off at $t = M_0/R$, as discussed above.

The derivative of $\ln(M_R + M_0 - Rt) = -R/(M_R + M_0 - Rt)$, so a quick comparison with the integral tells us that $-v_0 \ln(M_R + M_0 - Rt)$ is the solution.

Evaluating the limits yields: $v_F = -v_0 \ln[M_R + M_0 - R(M_0/R)] + v_0 \ln[M_R + M_0 - 0] =$
$v_0 [\ln(M_R + M_0) - \ln(M_R)] = v_0 \ln(1 + M_0/M_R)$.

That final expression, $v_F = v_0 \ln(1 + M_0/M_R)$, is known as the *Simple Rocket Equation*. Two points about it are noteworthy. First, the rate R at which the fuel is burned does not appear in it! As far as the ultimate velocity of the rocket is concerned, it does not matter if the force exerted by the fuel comes in a single blazing blast or in a series of gentle puffs spread out over a century. The total force is the same in either case.

Second, that logarithm is a terrible killer, if you are a rocket scientist. If your rocket consists of 50 tons of payload and 500 tons of fuel ($M_0 = 10M_R$), then your final speed will be $\ln(11) = 2.4$ times v_0. But if you decide you need super-speed, and so you build a stupendous mega-rocket which has the same 50 ton payload, but 50,000 tons of fuel, then your final speed will be $\ln(1001) = 6.9\ v_0$.

Humph. You burn 100 times as much fuel, and probably spend 500 times as much money building a huge rocket to carry all that fuel, and then what do you get? A paltry increase in speed from $2.4\ v_0$ to $6.9\ v_0$. The problem is, the larger you make your rocket then the more fuel you have to burn just to push the rest of the fuel. Our derivation was for a simple rocket, but you can toss in all the sophisticated break-away fuel tanks and multi-stage boosters you like and it won't help much. In the end, you will still run head-first into that logarithm.