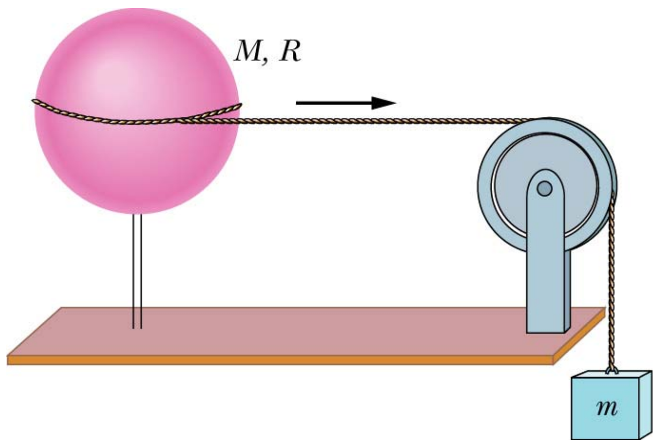
Physics 135-1 Sample Midterm II

1) A sphere of mass M = 5 kg and radius R = 40 cm has a cord tied around its equator, which is also attached to a small mass of m = 1.5 kg. The system is originally at rest. Then the small mass is released and allowed to fall. After it has fallen 50 cm, how fast (in radians per second) will the sphere be rotating? There is no friction in the system, and the pulley has no mass.



Solution

The gravitational potential released by the falling weight must equal the kinetic energies of the falling mass and the rotating sphere, so $mgh = \frac{1}{2} mv^2 + \frac{1}{2} I \omega^2$. We can relate v to ω with the formula $v = \omega R$, and I for a sphere is $\frac{2}{5} MR^2$. Thus $mgh = \frac{1}{2} m\omega^2 R^2 + \frac{1}{2} (\frac{2}{5} MR^2)\omega^2$. Inserting numbers: $(1.5)(9.8)(0.5) = (0.5)(1.5)(0.4)^2\omega^2 + (0.2)(5)(0.4)^2\omega^2$, or $\omega = (7.35 / 0.28)^{1/2} =$ 5.12 rad/s.

2) Suppose a ball of mass M = 0.5 kg and R = 3 cm is setting at the top of a ramp which has a height of 50 cm. Then, I release the ball and let it roll without sliping down the ramp. How fast will its center of mass be moving (horizontally) when the ball reaches the bottom of the ramp?

Solution: We can use conservation of energy, but we have to use it correctly. The ball will have both translational KE at the bottom of the ramp, given by $\frac{1}{2} mv^2$, and rotational KE, given by $\frac{1}{2} I\omega^2$. Fortunately, we know that $v = \omega R$ for a rolling object, so we can write the total KE as $\frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} Mv^2 + \frac{1}{2} (^2/_5 MR^2)(v/R)^2 = (\frac{1}{2} + {}^1/_5)Mv^2$. The gravitational potential energy of the ball is just E = Mgh, so we have $Mgh = (\frac{1}{2} + {}^1/_5)Mv^2$, or $v = (gh/0.7)^{1/2} = [(9.8)(0.5)/0.7]^{1/2} =$ 2.64 m/s.

3) Assume the Earth is a uniform sphere of radius 6370 km and mass 5.97×10^{24} kg. Assume it rotates exactly once every 24 hours. Then, suppose an asteroid (treat it as a point) with a mass of 1.83×10^{18} kg strikes the Earth along its equator, in the same direction as the Earth is turning, but angled at 60° to the radius of the Earth. The asteroid has a speed of 60 km/s relative to the Earth. By what amount will the period of the Earth speed up? (Hint – taking a ratio at the proper place could save you a lot of number-punching.)

Solution

An asteroid blasting into the Earth will generate an explosion – that is, it will generate a tremendous amount of heat energy. Thus kinetic energy is NOT conserved and we cannot go in that direction.

Instead we must conserve angular momentum. The angular momentum of the asteroid at the time of contact will be $L = r \times p = Rmv \sin(60°)$, where R is the radius of the Earth and mv is the momentum of the asteroid. The initial angular momentum of the Earth is $L = I\omega = ({}^2/_5\, MR^2)(2\pi/T)$, where T = (24 hrs)(60 min/hr)(60 s/min) = 86,400 s.

After the collision, the Earth and the asteroid will be rotating as one solid object. Their angular momentum will be $I_A\omega = I_A(2\pi/T_A)$, where T_A is the period we trying to calculate, and I_A is $({}^2/_5\, MR^2) + mR^2$, i.e., the moment of inertia of the Earth plus the moment of inertia of a small point mass "m" (the mass of the asteroid) on its surface.

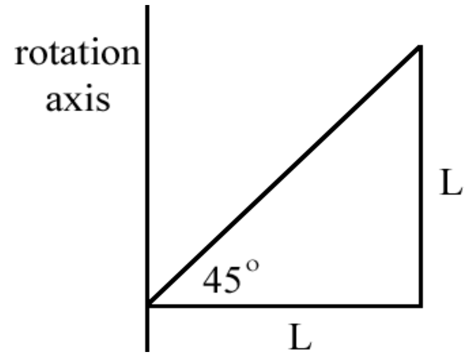
Equating before and after: $Rmv \sin(60°) + ({}^2/_5\, MR^2)(2\pi/T) = ({}^2/_5\, MR^2 + mR^2)(2\pi/T_A)$. We then divide through by R^2 to get $mv \sin(60°)/R + ({}^2/_5\, M)(2\pi/T) = ({}^2/_5\, M + m)(2\pi/T_A)$. Solving for T_A yields: $T_A = 2\pi({}^2/_5\, M + m)/[mv \sin(60°)/R + ({}^2/_5\, M)(2\pi/T)]$.

Inserting numbers for T_A :

$2\pi(2.388 \times 10^{24} + 1.83 \times 10^{18})/[(1.83 \times 10^{18})(6 \times 10^4)(0.866)/(6.37 \times 10^6) + (2.388 \times 10^{24})2\pi/86400]$
$= (1.5004258 \times 10^{25})/(1.4927722 \times 10^{16} + 1.7366026 \times 10^{20}) = 86{,}392.6$ s.

The period will decrease (Earth's rotation will speed up) by 7.4 sec.

4) Suppose you have a flat 45° triangle whose short sides are of length L. The tip of the 45° angle point is just touching a vertical rotation axis. The triangle has a total mass of M.

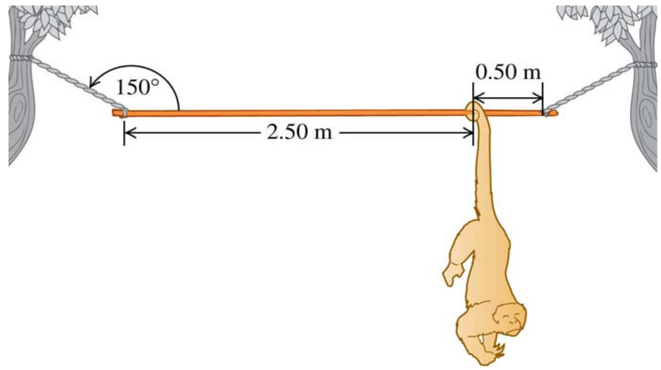


a) (8 points) Set up an integral to calculate the moment of inertia of the triangle around the rotation axis. It might be wise to explain your thinking, for partial credit.
b) (2 points) Evaluate the integral.

Solution
We need to evaluate $dI = r^2\, dm$. Let us cut the triangle into small vertical slices that are "y" high and dx wide. (Note that y is a variable.) We know that dm/M = dA/A, and in this case we have $dA = y\, dx$ and $A = ½\, L^2$. This yields $dm = My\, dx / ½\, L^2 = 2Mx\, dx / L^2$. We are able to substitute x for y because the 45° angle means that x = 0 when y = 0, and x = L when y = L, etc.

We now have $dI = x^2\, (2Mx\, dx / L^2) = (2M/L^2)x^3\, dx$. Integrating both sides gives us:
$I = (2M/L^2)(x^4 / 4) = (M/L^2)(x^4 / 2)$. Evaluation from 0 to L yields $½\, ML^2$.

5) A small monkey with a mass of M = 10 kg is hanging by his tail from a horizontal thin rod of mass m = 2 kg. The rod is 3 m long and the monkey is 0.5 m from the right-hand end, as shown at right. The rod in turn is suspended from a left rope that makes an 150° with the horizontal, and a right rope that is tilted at an unknown angle. What are the tensions in the left- and right-hand ropes, and what is the angle of the right-hand rope?



Solution: There are a number of ways to solve this problem. We will start by looking at the torque around the right end of the rod. The torque due to the monkey around this point is:
$\tau = rF \sin\theta = (0.5 \text{ m})(10 \text{ kg})(9.8 \text{ m/s}^2) \sin(90°) = 49$ N-m

The torque due to the CM of the rod around this point is:
$\tau = rF \sin\theta = (1.5 \text{ m})(2 \text{ kg})(9.8 \text{ m/s}^2) \sin(90°) = 29.4$ N-m

The torque due to the rope at the left end of the rod is given by:
$\tau = rF \sin\theta = (3 \text{ m})\, T \sin(30°) = 1.5\, T$
We thus have: 1.5 T = 49 + 29.4, or $T_{left} = 52.267$ N-m

Since the x-forces must add to zero, we know that $T_{left} \cos(30°) = T_{right} \cos\theta$. Since the y-forces must add to zero, we know that $T_{left} \sin(30°) + T_{right} \sin\theta = (10 \text{ kg} + 2 \text{ kg})(9.8) = 117.6$ N. A bit of algebra gives us: $T_{right} \cos\theta = 45.264$ and $T_{right} \sin\theta = 91.466$ N-m. We can then divide these two equations to get: $\tan\theta = 91.466 / 45.264 = 2.0207$, or $\theta = 63.67°$

We then have $T_{right} \cos(63.67) = 45.264$, or $T_{right} = 102.05$ N-m.

At this point we can double-check our answer by computing the torques around the *left* end of the rod. For the monkey, $\tau = (2.5)(10)(9.8) = 245$ N-m. For the CM of the rod, $\tau = (1.5)(2)(9.8) = 29.4$ N-m. For the right end of the rod, $\tau = (3)(102.05) \sin(63.67°) = 274.4$ N-m. We therefore have 245 + 29.4 = 274.4, which checks!