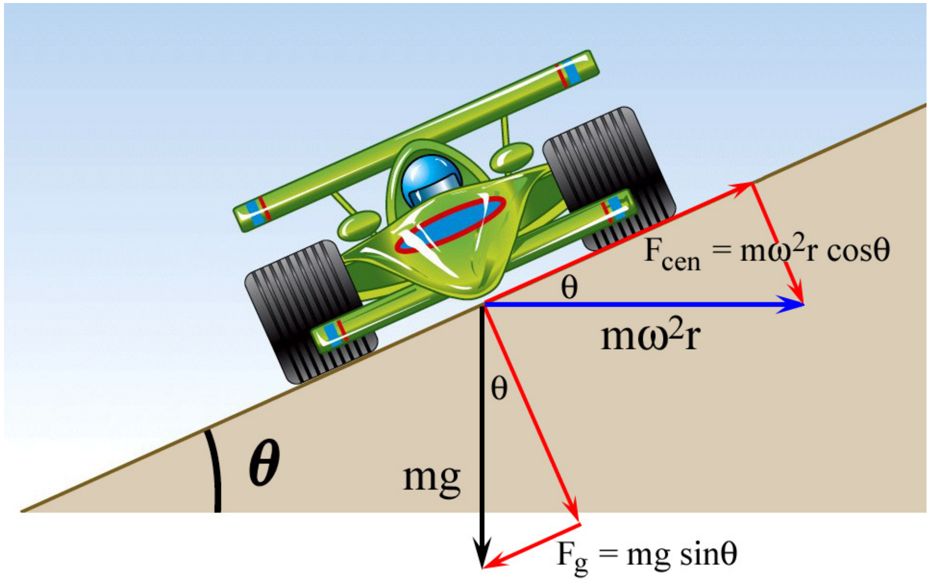
Race Car on a Track



Let's consider the motion of a race car as it goes around a circular, banked track. The question is, what is the top speed the car can achieve without sliding up and off the top of the embankment? The physical situation is shown in the illustration.

We have three forces to deal with: gravity, friction, and the so-called centrifugal force. Using the relation that $v = (2\pi r)f = r\omega$ for circular motion, where ω is the angular frequency, we can say that $F = m\omega^2 r$ for the "centrifugal force", and this is the force vector shown in blue in the illustration. Note that the "centrifugal force" must point directly away from the center of rotation, which in this case is the center of the track.

Rant – Talk about your awful terminology. There is no mythical "centrifugal force" pushing outward or upward on the car. An object moving in a circle must have an acceleration vector pointing *towards* the center of the circle, and something has to give the object a push or pull to make it turn inwards. In this case the inward acceleration can only be provided by the ramp. As it pushes the car to the center, by action-reaction the car pushes back. Hence the blue "centrifugal force" vector is shown coming *from* the car. It is a convenient fiction.

In truth, only gravity and friction are at work here. If the driver pushes the car's speed to the point where the acceleration needed to move it in a circle exceeds what gravity plus friction can supply, then the car's tires will "break loose" from the track and the car will more-or-less skid in a straight line, which will take it off the embankment. The car is not being "pushed" upward by any mythical force. Rather, it is not being *pulled enough* (by gravity and friction) to go in a circle of radius r at velocity v! So, it doesn't turn. Or at least, it doesn't turn enough, and hence it probably crashes spectacularly into the audience. We use the phrase "centrifugal force" because the magnitude of the car's acceleration must be $\omega^2 r = v^2/r$ if it stays on the circle, and if you multiply a by m then you have $F = ma = mv^2/r$, and that makes it a force. So to speak. So long as it stays on the circle.

Back to the car on the track. Calculating its maximum possible speed is a matter of balancing vectors. The force of gravity pulling the car down the slope, and the friction acting on the tires, must together be greater than or equal to mv^2/r (centrifugal force) or the car will begin to slide. Gravity is acting along the y-axis, but as usual for a ramp, only the vector component of the gravity acting down the slope can accelerate the car. The usual trigonometry gives us $F_G = mg \sin\theta$.

The centrifugal force vector can be treated exactly like any other force vector. The slope is at an angle to the force that the car is exerting on it, so we must resolve the force into components. An examination of the illustration shows that the component of the centrifugal force acting along the ramp is $F_{cen} = m\omega^2 r \cos\theta$.

Next, let us look at the friction. We have $F_f = \mu N$. The normal force N is the force acting directly into the slope, and in this case it is being generated by two sources: gravity and the centrifugal force. Contemplation of the illustration and some basic trig tells us that the normal component due to gravity is $mg\cos\theta$, and the normal component due to centrifugal force is $m\omega^2 r\sin\theta$.

The car will begin to skid when the centrifugal force needed to make it turn along the track *just* barely equals gravity and friction. We have:

$m\omega^2 r\cos\theta$	$=$	$mg\sin\theta$	$+$	μ [$mg\cos\theta$	$+$	$m\omega^2 r\sin\theta$]
(sloped component of centrifugal force)		(sloped component of gravity)		(normal component of gravity		(normal component of centrifugal force)

As it often does in problems that involve a single mass plus friction, gravity, or centrifugal force, the mass cancels out. (The size of the car doesn't matter, only how good its tires are.) A bit of algebra gives us the final answer: $v^2 = gr(\mu + \tan\theta) / (1 - \mu\tan\theta)$

We can also ask the opposite question: what is the *minimum* speed the race car must go so that it will not slide *down* the track? There are two answers to this. If the coefficient of friction is high enough, then the minimum speed is zero because the friction is greater than gravity. The car will just set on the track, unmoving.

More amusingly, if the slope is steep enough, we have:

$m\omega^2 r\cos\theta$	$=$	$mg\sin\theta$	$-$	μ [$mg\cos\theta$	$+$	$m\omega^2 r\sin\theta$]
(sloped component of centrifugal force)		(sloped component of gravity)		(normal component of gravity		(normal component of centrifugal force)

This is the same as our previous answer, except that we have reversed the sign of the friction. Since friction always opposes the overall acceleration, we must give it one sign if the car is sliding upwards and the opposite sign if the car is sliding downwards.