Physics 335-0, Homework #1

The assignment is to calculate what percent of their total luminosity that various objects emit as visible light. Most stars can be assumed to be blackbodies (so long as you don't worry *too* much about very high-energy X-rays or very low-energy radio waves, which can be produced in the star's atmosphere by magnetic storms). So, we can start with the Planck distribution:

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/kT\lambda} - 1} \qquad \text{(Planck Blackbody Formula)}$$

This formula tells us the intensity **I** of the radiation emitted (per second per unit area) at a specific wavelength λ and temperature T. We need to integrate this function (at constant T) from λ = 400 nm to 700 nm, which correspond to very red and very violet light, respectively .

It is convenient to do some algebra before we plow ahead to the actual calculation. We will set

$$x = \frac{hc}{kT\lambda}, \text{ which therefore means } \lambda = \frac{hc}{kTx}, \text{ and } d\lambda = -\frac{hc}{kT}\frac{dx}{x^2}$$

These substitutions turn $I(\lambda,T)\,d\lambda$ into: $-2\pi hc^2\left(\frac{k^5T^5x^5}{h^5c^5}\right)\frac{1}{e^x - 1}\frac{hc}{kT}\frac{dx}{x^2} = -2\pi\left(\frac{k^4T^4}{h^3c^2}\right)\frac{x^3}{e^x - 1}dx$

AMUSING MATH SIDEBAR

To find the total power emitted by the blackbody ($\lambda = 0$ to ∞), all you need do is integrate this function from $x = \infty$ to 0. We have:

$$P = -2\pi\left(\frac{k^4T^4}{h^3c^2}\right)\int_\infty^0 \frac{x^3}{e^x - 1}dx = \left(\frac{2\pi k^4}{h^3c^2}\right)T^4\int_0^\infty \frac{x^3}{e^x - 1}dx$$

The indefinite integral cannot be calculated in closed form, but fortunately the definite integral can be evaluated, and it turns out to be $\pi^4/15$. The final expression for the power is thus:

$$P = \left(\frac{2\pi k^4}{h^3c^2}\right)T^4\frac{\pi^4}{15} = \left(\frac{2\pi^5k^4}{15h^3c^2}\right)T^4 = \sigma T^4$$

σ is the Stefan-Boltzmann constant, and inserting values for k, h, and c gives $\sigma = 5.67 \times 10^{-8}$ W / m^2 K^4. You might want to work through the units in that last bracket and convince yourself that σ really does have the units of *power* per meter squared per K^4. $P = \sigma T^4$ only tells you how much energy the blackbody radiates per second per unit area – which is a rate. To find the total energy emitted by a blackbody, you must multiply by the time it shines and by the total surface area of the blackbody.

BACK TO THE HOMEWORK PROBLEM

In our case, we want to know more than just the total power emitted per second per area. We also want to know the *percentage* of the total power emitted within a certain frequency range, i.e., that of visible light. Therefore, we need to divide the Planck integral by the power emitted by the star, i.e., we need to calculate $I(\lambda,T) / \sigma T^4$.

Dividing the indefinite integral $x^3 / (e^x - 1)$ by σT^4 and remembering that $x = hc / kT\lambda$, and cancelling out a lot of constants, finally gives us:

$$P = \left(\frac{15}{\pi^4}\right)\int_{x_L}^{x_h} \frac{x^3}{e^x - 1}\,dx\,, \text{ where } x_h = \frac{hc}{kT\lambda} = \frac{hc}{k(400\text{nm})}T^{-1} = \frac{35{,}968}{T}\,,$$

$$\text{and likewise } x_L = \frac{hc}{k(700\text{nm})}T^{-1} = \frac{20{,}553}{T}\,.$$

x_h and x_L indicate x “high” and “low”, of course, which in turn correspond to the 400 nm and 700 nm limits that we want to put on λ. Note that “x” is dimensionless, thus 35,968/T and 20,553/T are dimensionless as long as T is in degrees Kelvin. (T is the temperature of the star.) As mentioned above, this integral cannot be evaluated in closed form, so you will have to crunch through it numerically. There are a myriad of ways to do this. You can turn to Mathematica or Matlab, you can use a programmable calculator, or you can just use a spreadsheet. It doesn’t matter how you do it. However, since you need to find the integral for several values of the temperature, you should program it in such a way that you can just enter the temperature and hit the return key to find the efficiency. Briefly note your numerical method in your solution (details on how it was coded are not necessary), and indicate how you convinced yourself that the routine was correct and accurate enough for our purposes.

Here are the objects and their temperatures in K°:

The Sun	5,780
Sirius A (brightest star in Earth’s sky)	9,860
Sirius B (tiny companion to Sirius A)	24,800
Procyon (bright nearby star)	6,550
Alpha Centauri B (nearby roughly solar-size star)	5,290
Proxima (closest star to the Sun)	3,100
Incandescent Light Bulb	2,900
High-Intensity Halogen Bulb	3,300

The object is simply to calculate the percentage of their power that these various blackbodies emit as visible light, using the integral at the top of this page.

When you are finished, look at the results for all the stars. Do you notice anything special about the Sun and Procyon?

For the light bulbs, how does the incandescent bulb compare to the Sun in terms of energy efficiency? How does it compare to a halogen bulb?