Thursday, September 18, 2003

Quantum Mechanics

Choose 3 out of 4 problems

1. (a) A quantum system is described by the Hamiltonian

$$H = H_0 + \lambda H_1(t)$$

where λ is a small constant and H_0 is independent of time. The eigenfunctions and eigenvalues of H_0 are ϕ_n, ϵ_n where $n = 1, 2, 3, ...$ and ϕ_1 is non-degenerate. At initial time $t = t_1$ the wave function is ϕ_1. Find an expression, correct to lowest non-trivial order in λ, for the probability that at time $t_2 > t_1$, the system will have made a transition to the state $\phi = \phi_n$ where $n \neq 1$.

(b) Consider a spin $\sigma = \frac{1}{2}\hbar$ system, the magnetic dipole with dipole moment $\vec{\mu} = \mu_0\vec{\sigma}$. For $t < 0$ it is immersed in a uniform magnetic field $\vec{B_0} = B_0\hat{z}$ and is known to be in spin eigenstate $\chi_{+1/2}$. At time $t = 0$ an additional rotating magnetic field is turned on with

$$\vec{B_1} = B_1[\cos(2\omega_0 t)\hat{x} - \sin(2\omega_0 t)\hat{y}],$$

where $\omega_0 = \frac{\mu_0 B_0}{\hbar}$. Find the spin wave function of the system for any $t > 0$.

(c) How long does it take to flip the spin to $\chi_{-\frac{1}{2}}$ for the first time?

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2. A particle of mass m is bound to the origin by a spherically symmetric linear restoring force. The energy levels are equally spaced at intervals $\hbar\omega_0$ above the ground state energy, $E_0 = \frac{3}{2}\hbar\omega_0$.

   (a) In a cartesian coordinate system, tabulate the occupation numbers for the various states of the oscillators for the ground and first 3 (different) excited energy levels. Determine the degeneracy of these levels.

   (b) Find a general formula as a function of n for the degeneracy of the states with energy $E_n = (n + \frac{3}{2})\hbar\omega_0$.

   (c) In spherical coordinates, write down (DO NOT SOLVE) the differential equation for the radial part $R(r)$ of the eigenstate with quantum numbers n, ℓ, m if the full wave function is $\Psi(r, \theta, \phi) = R(r)Y_{\ell,m}(\theta, \phi)$. *Useful information:*

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi) - \frac{L^2}{r^2}\Psi$$

   where L^2 is the operator of total orbital angular momentum (in integer units).

   (d) What values of ℓ are allowed for a state with quantum number n.

   (e) Let ℓ be the angular momentum quantum number such that $< Y_\ell|L^2|Y_\ell >= \hbar^2\ell(\ell+1)$. Find the radial dependence of $R(r)$ for $r \to 0$(easier) and $r \to \infty$(not as easy) for all n and ℓ.

3. (a) Define

$$J = \frac{\hbar i}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$

   If you interpret J as the 1-dimensional current, derive the expression for the conservation of probability in a 1-D system.

   (b) Consider a 1-dimensional potential barrier of height $+V_0$ and thickness L. Write down the boundary conditions for the wave functions in the 3 regions - you do not need to solve for the coefficients.

   (c) Derive the condition for 100% transmission across the barrier.

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4. (a) Calculate to within a multiplicative constant the differential elastic scattering cross section for neutron-neutron scattering, using the Born approximation and assuming that the interaction potential is $-V_0$ for the spherical radial coordinate $r \leq a$ and 0 for $r > a$.

   (b) Show that similar energy and angle dependence will be obtained for differential cross sections for any finite range potential at 'low' momentum. Provide a definition of low momentum.

   (c) Write down the condition under which the Born approximation is reasonable for this potential for **all momenta.**