Department of Physics & Astronomy Qualifying Exam
Classical and Statistical Mechanics

Problem 1 (Stat. Mech)

For a gas of molecules of mass m, the chemical potential μ, the temperature T , and the number density, n, are related by

$$e^{\mu/k_B T} = n\left(\frac{2\pi\hbar^2}{mk_B T}\right)^{3/2}\frac{1}{z_{rv}(T)}$$

where $z_{rv}(T)$ is the roto-vibrational partition function for one molecule.

The goal in this problem is to apply this formula to understand molecular H_2 gas. For H_2,

$$T_{rot} \equiv \frac{\hbar^2}{2k_B I} = 85\ \text{K} \quad (\text{I = moment of inertia of } H_2).$$

$$T_{vib} \equiv \frac{\hbar\omega}{k_B} = 6215\ \text{K} \quad (\omega = \text{vibrational frequency of } H_2).$$

(a) H_2 comes in two forms, ortho – H_2 and para – H_2. In ortho – H_2, the protons' spins are in the $S_{tot} = 1$ state, while in para – H_2, they are in the $S_{tot} = 0$ state. Evaluate z_{rv} for ortho – H_2 and para – H_2.

(b) Show that in equilibrium at a temperature T_0 such that $T_{vib} >> T_0 >> T_{rot}$,

$$N_{ortho} : N_{para} = 3:1$$

[It is not enough to say that there are three states (m = 0, ± 1) for $S_{tot} = 1$]

(c) A sample of H_2 is cooled rapidly from T_0 to T = 10 K, so the ratio N_{ortho}/N_{para} is still 3. Show that the system is now not in equilibrium, and state, with reasons, in which direction the ortho $\rightleftharpoons$ para conversion will proceed.

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Problem 2 (Stat. Mech)

A substance has the entropy function

$$S = a\ N^{1/4} V^{1/4} E^{1/2}$$

where a is a constant, and N, V, and E are the particle number, volume, and total energy.

A cylinder contains a partition that is initially <u>fixed</u>, dividing it into two equal halves, each with volume $\frac{1}{2}V_0$. $\frac{1}{5}N_0$ molecules of the above substance, with energy $\frac{1}{6}E_0$, are introduced into the left half, and $\frac{4}{5}N_0$ molecules, with energy $\frac{5}{6}E_0$, are introduced into the right half.

(a) After equilibrium is reached, what is the energy in each half? (The partition carries heat but it does not move.)

(b) If the partition is also allowed to move, what are the equilibrium volumes and energies of each side?

(c) Find the appropriate free energy which is a thermodynamic potential when expressed in terms of N, p (pressure), and T (temperature). Express the free energy in terms of these variables.

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Problem 3 (Stat. Mech)

(a) Consider a dense gas of non-interacting ultrarelativistic fermions, assuming that the chemical potential is negligible compared to k T. Calculate the number density, pressure and energy density of such a gas. You need not evaluate dimensionless integrals of order unity.

(b) If the gas expands adiabatically while remaining in equilibrium, how do the number density and energy density depend on the volume of the system?

The above approximation apply to the electrons and positrons in the very early stages of the universe, when the temperature $T \sim 10^{11}$ K.

(c) Is the assumption made in (a) that the particles are non-interacting reasonable? Why? (Hint: Estimate the average Coulomb interaction energy?)

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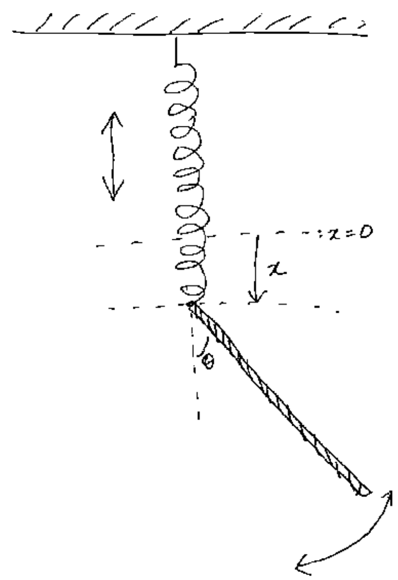
Problem 4 (Class. Mech)

A uniform bar of mass M and length $2l$ is suspended from one end by a spring of force constant k in the earth's gravitational field. The end of the spring can move ONLY in the vertical direction (it does not swing). The bar can swing only in one vertical plane (there are no rotations about a vertical axis). At equilibrium $x = 0$ and $\theta = 0$.



(a) Determine the equations of motion.

(b) To LOWEST order in x and θ, what are the solutions to these equations?

(c) To next-lowest order, describe qualitatively the behavior of the system.

(d) Solve the equations, given that at $t = 0$, $\theta = \theta_0$ ($\neq 0$), $d\theta/dt = 0$, $x = 0$, $dx/dt = 0$.

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Problem 5 (Class. Mech)

Consider a particle that has a nearly circular orbit in which the central force is described as $F(r) = -k/r^n$ (where n is an integer and $k > 0$).

(a) Derive the equation of motion of the particle relative to a circular orbit of radius r_0 and describe its motion. Assume that $r = r_0 + x$, where $x << r_0$.

(b) Determine the apsidal angle and describe the conditions of n for which the orbits are closed (assuming that we exclude values of n equal to or smaller than –6).

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Problem 6 (Class. Mech)

A rotating rigid body has a summed kinetic energy

$$T_{rot} = \frac{1}{2}\int d^3x\ \rho(\vec{r})\ \ (\vec{\omega}\times\vec{r})^2$$

(a) Show this can be written:

$$T_{rot} = \frac{1}{2}\sum_{ij} I_{ij}\,\omega_i\omega_j$$

where $I_{ij} \equiv$ Inertia Tensor

$$= \int d^3r\ \ \rho(\vec{r})\left[\vec{r}^{\,2}\delta_{ij} - x_i x_j\right]$$

What is the form of I_{ij} in the principal axis frame of reference?

(b) Consider a thin, flat rigid body of arbitrary shape. Let the z axis be $\perp$ to the flat surface.

Prove that $I_x + I_y = I_z$

(c) Consider a thin homogeneous flat circular disk of radius R . Find its principal moments of inertia.

(d) Suppose this disk is placed on edge upon a flat surface. It will fall over. What is the velocity of the center of mass when it hits the flat surface?