Physics Qualifier Problems
Quantum Mechanics

September 17, 2004

Instructions:

Four problems are stated on these pages.

Solve **3** of these problems.

Please indicate which problems you want graded for credit.

Do NOT write your name on the blue book – write your *code.*

Write all your answers in blue books. These sheets stating the problems will be collected and discarded at the end of the exam period.

You are allowed to use the single index card you prepared in advance.
You can refer to the mathematic reference book in the room.
You are NOT allowed to use notes or books, and calculators are not allowed.

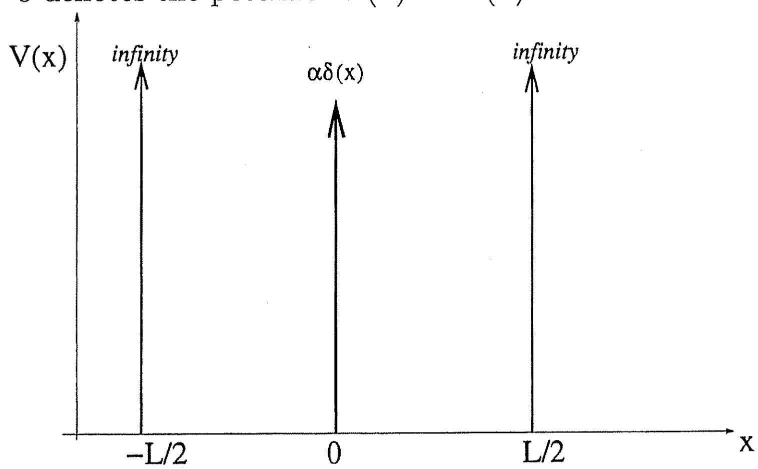
Good Luck!

(1) (a) You are given the one-dimensional δ-function potential $V(x) = \alpha\delta(x)$, where α is a positive real constant.
Show that the effect of this potential on the wave function of a particle is such that the amplitude of the wave function is continuous at $x = 0$, but that the slope changes according to the relation

$$\psi'(+\epsilon) = \psi'(-\epsilon) - \frac{2m\alpha}{\hbar^2}\,\psi(0)$$

where ϵ is an infinitesmal and m is the mass of the particle.

(b) Suppose you are now given the potential shown in the figure below, where the spike at $x = 0$ denotes the potential $V(x) = \alpha\delta(x)$ where α is a constant.



Show that the wave functions resulting from this potential fall into two classes, even and odd, and write appropriate expressions for the wave functions in each of these classes in terms of the wave vector k for a free particle (which is related to the energy as $E = \frac{\hbar^2 k^2}{2m}$ where m is the particle mass); sketch these wave functions.

(c) Show that the energy levels for the odd solutions follow from the condition

$$k = \frac{2n\pi}{L}$$

where $n = 1, 2, \ldots.$

(d) Show that the allowed values of k in the limit $\alpha \to 0$ for the even solutions are given by

$$k = \frac{(2n+1)\pi}{L}$$

and that for the general case of $\alpha \neq 0$, the energy levels follow from the solutions of the equation

$$k\frac{\cos\left(k\frac{L}{2}\right)}{\sin\left(k\frac{L}{2}\right)} = -\frac{m\alpha}{\hbar^2}.$$

(2) (a) The expression for the energy shift ΔE_n of a system in state ψ_n resulting from a perturbation $\hat{H}'$ in first order is

$$\Delta E_n = \langle \psi_n | \hat{H}' | \psi_n \rangle .$$

Use this expression to prove the following theorem (called the 'Hellman-Feynman Theorem'):

$$\frac{\partial E_n(\lambda)}{\partial \lambda} = \langle \psi_n(\lambda) | \frac{\partial \hat{H}(\lambda)}{\partial \lambda} | \psi_n(\lambda) \rangle$$

where λ is a parameter measuring the strength of a perturbation in the Hamiltonian (*i.e.*, $H = H(\lambda)$) and we assume that $\psi_n(\lambda)$ is normalized and non-degenerate; note this relation is exact.

(b) The radial part of the Hamiltonian for the hydrogen atom is given by

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} - \frac{e^2}{r}$$

and the corresponding energy levels are given by

$$E_n = -\frac{R}{n^2} \qquad \text{with} \qquad R = \frac{me^4}{2\hbar^2}.$$

Using the Hellman-Feynman theorem (you may use the result of (a) irrespective of whether you can derive it), and defining $\lambda \equiv e^2$, show that

$$\left\langle \frac{1}{r} \right\rangle = \frac{1}{a_0\, n^2}$$

where $a_0 = \hbar^2/(me^2)$ is the Bohr radius.

(c) Rewrite the energy of the hydrogen atom in terms of the radial quantum number n_r (which you will keep constant) and the angular momentum quantum number ℓ, where $n = n_r + \ell + 1$, so that the energy becomes

$$E_n = -\frac{R}{(n_r + \ell + 1)^2}.$$

Defining now $\lambda = \ell$, show that

$$\left\langle \frac{1}{r^2} \right\rangle = \frac{1}{a_0^2\, n^3\, (\ell + \frac{1}{2})}.$$

(3) A particle resides in the ground state of a one-dimensional infinite square-well potential of the form

$$V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}$$

(a) Calculate the eigenvalues and normalized eigenfunctions for this potential.

(b) At $t = 0$ an additional potential

$$V(x) = \alpha V_0 \, \delta\left(x - \frac{a}{4}\right)$$

is applied, and remains there for a time T after which it is removed. Calculate the time-dependent amplitude and probability that the particle will be in an excited state for times greater than T using first-order time-dependent perturbation theory.

(c) Are there some excited states for which there are no transitions (in lowest order), and if so, what is the criterion?

(4) Consider two non-interacting particles with coordinates x_1 and x_2 residing in a one-dimensional harmonic potential. The particles have mass m and are *identical*; the frequency of oscillation is ω. The normalized single-particle wave functions are $\psi_n(x)$, specifically:

$$\psi_n(x) = C_n\, H_n(u)\, \exp(-u^2/2)$$

where $u \equiv bx$ (with $b \equiv \sqrt{m\omega/\hbar}$), and the normalization constant is $C_n \equiv (1/\sqrt{2^n\, n!})\, \sqrt{b/\sqrt{\pi}}$. The first few *Hermite polynomials* are

$$H_0(u) = 1 \qquad H_1(u) = 2u \qquad H_2(u) = -2 + 4u^2 \qquad H_3(u) = -12u + 8u^3.$$

The two particles have parallel spins.

First, assume the particles are bosons.

(a) Write down the correctly symmetrized ground state, $\Psi_B(x_1, x_2)$, given the single-particle wave functions $\psi_n(x)$ above. What is the energy of this ground state?

(b) What is the most probable value for the separation, $D \equiv u_2 - u_1$? Sketch the probability as a function of D. What is the root-mean-square separation?

Now assume the particles are fermions.

(c) Write down the correctly symmetrized ground state, $\Psi_F(x_1, x_2)$, What is the energy of this ground state?

(d) What is the most probable value for the separation? Sketch the probability as a function of D. What is the root-mean-square separation?

Hint: It will be useful to use the variable transformation

$$D \equiv u_2 - u_1 \qquad \text{and} \qquad S \equiv u_2 + u_1.$$

Recall: The r.m.s. of a quantity A is $\sqrt{\langle (A - \langle A \rangle)^2 \rangle}$.

For reference:

$$\int_{-\infty}^{\infty} du\, \exp(-u^2/2) = \sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} du\, u\, \exp(-u^2/2) = 0$$

$$\int_{-\infty}^{\infty} du\, u^2\, \exp(-u^2/2) = \sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} du\, u^4\, \exp(-u^2/2) = 3\sqrt{2\pi}$$