Classical and Statistical Mechanics

This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the exam for a total of four problems.

Classical Mechanics - do 2 out of 3 problems

1. Consider a particle of mass m moving in a three dimensional harmonic oscillator potential,

$$V(r) = \frac{1}{2}kr^2.$$

   (a) Derive the Lagrangian and the corresponding equations of motion.

   (b) Show that, in general, the particle's orbit is an ellipse centered at $r = 0$.

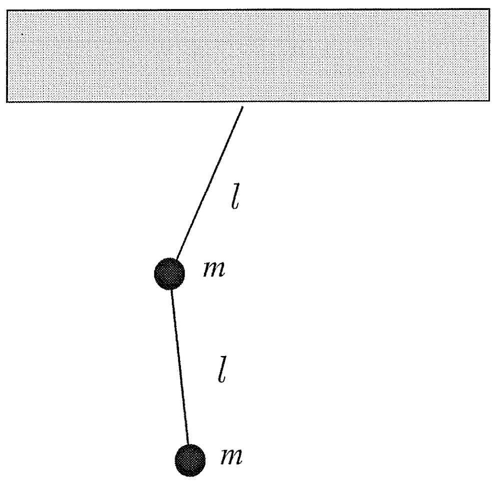
   (c) Relate the parameters of the ellipse to the particle's angular momentum, L, and energy, E.

2. Consider a simple pendulum consisting of a bob of mass m suspended by a massless rod of length l. From the bob of this pendulum is suspended a second identical pendulum. Both pendulum are confined to move in the same plane.

   (a) Find Lagrange's equations of motion for the system.

   (b) Consider the case of small oscillations and calculate the characteristic frequencies.

   (c) Find the normal modes of the system.



3. A uniform rod of mass m and length l is initially positioned vertically with its lower end in contact with a frictionless horizontal table. The rod is given an infinitesimal angular displacement from its initial position, so that it falls over. When the rod makes an angle ϕ with respect to the vertical, find:

   (a) the angular speed of the rod.

   (b) the speed at which the center of the rod is falling.

   (c) the speed at which the lower end of the rod is moving.

Statistical Mechanics - do 2 out of 3 problems

1. A weight W is suspended from the end of a rubber band at absolute temperature T. Assume as a simple microscopic model of the rubber band that it consists of a linked polymer chain of N segments joined end to end; each segment has length a and can be oriented either parallel or antiparallel to the vertical direction.

   (Neglect the kinetic energies or weights of the segments themselves, and any interactions between the segments).

   (a) Making use of the fact that flipping a link from down to up requires work, write down the single link partition function. For distinguishable links, what is the N link partition function?

   (b) Determine the rubber band's mean energy.

   (c) Find an expression for the resultant mean length $\bar{l}$ of the rubber band as a function of W, T, a, and N.

2. Each Hemoglobin molecule in our blood has four absorption sites, with each site capable of absorbing an O_2 molecule. The binding energy is 0.7 eV. In our lungs, hemoglobin molecules come into contact with air at a pressure of one atmosphere and (body) temperature of about 310 K. In this problem, you will calculate the probability that any given absorption site is occupied by an O_2 molecule.

   (a) Starting from the quantum states for a particle of mass m in a 3-d box of dimensions L_x, L_y and L_z, which have energy

   $$E(n_x, n_y, n_z) = (h^2/8m)[(n_x/L_x)^2 + (n_y/L_y)^2 + (n_z/L_z)^2],$$

   show that the partition function of a single point particle of mass m in a box of volume V is given by $Z_1 = (V/v_q)$, where

   $$v_q = (\frac{h^2}{2\pi mkT})^{3/2}$$

   . Work in the high temperature limit.

   (b) Use your result from part (a) to find the Helmholz free energy of an ideal gas consisting of N identical monatomic particles, as a function of V, T, N, and m. (You may need Stirling's approximation: $ln(N!) = Nln(N) - N$).

   (c) Now consider the Hemoglobin absorption sites discussed above. At equilibrium, what fraction of these sites will be occupied by an O_2 molecule? To simplify the calculation, treat the O_2 molecules in your lungs as monatomic particles of mass 32 m_p . Note that air is basically made up of 20% O_2 and 80% N_2. Also, note the following conversion factor: 1 atm $\approx 10^5 N/m^2$.

3. In an absorption refrigerator, heat from a gas flame at temperature T_f provides the energy required for operation. More specifically, the operation is as follows: Heat is absorbed from the flame (that is at temperature T_f), heat is absorbed from the inside of the refrigerator (temperature T_i), and waste heat is expelled into the room (temperature T_r). The efficiency of the absorption refrigerator is defined as the heat extracted from the inside of the refrigerator divided by the energy (heat) supplied by the flame. Derive an expression for the maximum possible efficiency, as a function of T_f, T_i, and T_r? You do *not* need to know the details of the operation of the refrigerator to solve this problem. But one could, for example, use the flame to drive a steam engine that would power the refrigerator's compressor.

   (HINT: Note that if the refrigerator is to run continuously, then the refrigerator mechanism itself must have no long-term changes in state, and thus its entropy must not change. Using the second law of thermodynamics , we can therefore conclude that the total entropy of the three heat reservoirs cannot decrease.)