Quantum Mechanics

Choose 3 out of 4 problems

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1. A box containing a particle is divided into two compartments by a thin partition. If the particle is known to be in the left compartment with certainty, then the state vector representing the position eigenstate is $|L\rangle$, and similarly if the particle is known to be in the right compartment with certainty then the position eigenvector is $|R\rangle$. Classically a particle on the left side cannot pass through the barrier to the right side. However, a quantum particle can tunnel through the barrier from one side to the other. Then the most general state vector can be written as

$$|\psi\rangle = \psi_R|R\rangle + \psi_L|L\rangle\,, \tag{1}$$

where ψ_R and ψ_L can be regarded as wave functions for the particle to be in the right or left compartment (here you can neglect spatial variations of the wave function within each compartment). The Hamiltonian describing the tunnelling effect is

$$\widehat{H} = \Delta\Big(|L\rangle\langle R| + |R\rangle\langle L|\Big) \tag{2}$$

where Δ is a real number with the dimension of energy.

(a) Find the normalized energy eigenvectors and the corresponding energy eigenvalues.

(b) In the Schrödinger picture the base kets $|R\rangle$ and $|L\rangle$ are fixed, and the state vector evolves in time. Suppose the state of the particle is given by $|\psi\rangle$ at time $t = 0$. Find the state vector at time $t > 0$ by applying the time-evolution operator.

(c) Suppose at $t = 0$ the particle is on the right side of the partition with certainty. What is the probability of observing the particle on the left side of the partition as a function of time?

(d) Write down the coupled Schrödinger equations for the wave functions $\psi_R(t)$ and $\psi_L(t)$. Show that the solutions to the coupled Schrödinger equations are just what you expect from part (b).

2. The ground state of Hydrogen is four-fold degenerate when we consider the possible spin configurations of the electron and the proton but neglect the "hyperfine" interaction. The hyperfine Hamiltonian is obtained from the interaction energy of two point magnetic dipoles,

$$\mathcal{H}_{hf} = \frac{\boldsymbol{\mu}_p \cdot \boldsymbol{\mu}_e}{r^3} - \frac{3(\boldsymbol{\mu}_p \cdot \mathbf{r})(\boldsymbol{\mu}_e \cdot \mathbf{r})}{r^5} - \frac{8\pi}{3}\boldsymbol{\mu}_p \cdot \boldsymbol{\mu}_e \delta(\mathbf{r})$$

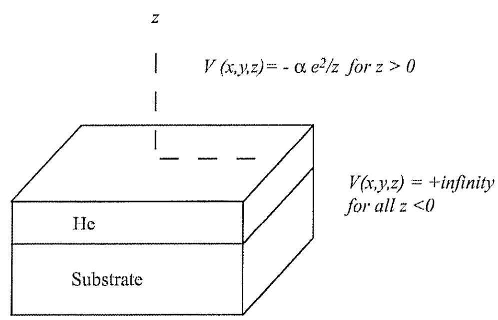
where $\mathbf{r}$ is the position operator for the electron relative to the proton, and $\boldsymbol{\mu}_e$ and $\boldsymbol{\mu}_p$ are the electron and proton magnetic moments. The latter are related to the intrinsic spin angular momenta of the electron and proton by $\boldsymbol{\mu}_e = -\frac{e}{m_e c}\mathbf{S}_e$ and $\boldsymbol{\mu}_p = g_p \frac{e}{2M_p c}\mathbf{S}_p$, where m_e and M_p are the masses for the electron and proton and $g_p \simeq 5.6$ is the g-factor for the proton.

(a) What is the ionization energy of Hydrogen? Express your result in terms of the fine structure constant, $\alpha = e^2/\hbar c$, and the rest mass energy of the electron as well as in units of eV.

(b) Calculate the hyperfine energy level shifts for the four spin configurations of the proton and electron in the $1s$ orbital. Express your result in terms of the Rydberg, the fine structure constant and other relevant parameters. Specify the quantum numbers and wave functions for each state.

(c) Calculate the hyperfine splitting between the ground state and the first excited state. What is the corresponding wavelength of radiation associated with a transition between these levels?

3. The force between an electron (mass m and charge e) and a dielectric film of liquid Helium is given by a potential, $V(z) = -\alpha e^2/z$, for $z > 0$, where $0 < \alpha < 1$ is related to the dielectric properties of the Helium film and the substrate on which the film resides (see Figure). In this example assume $\alpha = 0.01$. For $z \leq 0$ assume the potential is infinitely repulsive, thus preventing penetration of the electron into the Helium film.



(a) Write down the Hamiltonian for the electron in the half-space $z > 0$. What conservation laws are obeyed for the motion of the electron?

(b) Write down the wave equation for the stationary states of the electron. Use the conservation laws to reduce the stationary-state wave functions to the form, $\psi(\mathbf{r}) = \chi(x, y)\phi(z)$. Obtain the solutions for $\chi(x, y)$ and a differential equation for $\phi(z)$, the amplitude for the electron to be a distance z above the film. What are the quantum numbers that define the wave functions $\chi(x, y)$ for motion in the plane of the film?

(c) What are the conditions on $\phi(z)$ for bound state solutions for the electron on the surface? Use these two conditions to construct the ground-state wave function for the electron bound to the Helium surface. Hint: You may safely assume that the ground-state wave function is the simplest wave function obeying the boundary conditions at $z = 0$ and $z \to \infty$ that has no zeroes for $0 < z < \infty$.

(d) What is the most probable position of the electron (in its ground state) above the film? Give your answer in terms of physical constants: $\alpha, m, \hbar, c, e$, etc. and in terms of Å. What is the binding energy for the electron in the ground state? Give your answer in terms of physical constants: $\alpha, m, \hbar, c, e$, etc. and in terms of eV. What is the threshold wavelength of the radiation required to photo-ionize the ground-state electron from the film?

4. An electron and its anti-particle, the positron, can form a hydrogenic "atom" called positronium that lives sufficiently long to observe radiative transitions.

   (a) What is the most probable distance between the electron and positron in the $1s$ orbital?

   (b) What is the ionization energy of positronium?

   (c) Write down Fermi's Golden rule and the relevant matrix element for the transition probability for the spontaneous decay of the $2p$ level of *positronium* to the $1s$ level via photon emission. Be sure to specify the initial and final states of the radiation field as well as that of the atom. The photon field is described by vector potential,

$$\mathbf{A} = \sum_{\mathbf{k}\lambda} \sqrt{\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}}}} \left\{ a_{\mathbf{k}\lambda}\, \varepsilon_{\mathbf{k}\lambda}\, e^{i\mathbf{k}\cdot\mathbf{r}} + a^{\dagger}_{\mathbf{k}\lambda}\, \varepsilon^{*}_{\mathbf{k}\lambda}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \right\} , \qquad (3)$$

   where $a^{\dagger}_{\mathbf{k}\lambda}$ $(a_{\mathbf{k}\lambda})$ are the raising and lower operators for the quanta of the radiation field, $\varepsilon_{\mathbf{k}\lambda}$ is the direction of the electric field for radiation in mode with wavevector $\mathbf{k}$ and one of two possible polarization states labelled by λ, and $\omega_{\mathbf{k}} = c|\mathbf{k}|$ is the dispersion relation for photons travelling at speed c. What is the dominant decay channel, electric dipole, magnetic dipole, electric quadrupole, magnetic quadrupole, etc.?

   (d) Given that the same transition for hydrogen has a lifetime of $1.6 \times 10^{-9}\, sec$ calculate the lifetime for the $2p \rightarrow 1s$ transition in positronium.