Physics Qualifier Problems
Electricity and Magnetism

September 15, 2005

Instructions:

Four problems are stated on these pages.

Solve **3** of these problems.

Please indicate which problems you want graded for credit.

Do NOT write your name on the blue book – write your *code*.

Write your answers for each problem in a separate blue book. These sheets stating the problems will be collected and discarded at the end of the exam period.

You are allowed to use the single index card you prepared in advance.
You can refer to the mathematic reference book in the room.
You are NOT allowed to use notes or books, and calculators are not allowed.

Good Luck!

(1) **Charged Pion Decay** – The charged pion π^+ decays almost 100% of the time into a muon μ^+ and a neutrino ν. ($\pi^+ \to \mu^+ + \nu$)

(a) Calculate, in the rest-frame of the pion, the muon energy as a function of the pion mass m_π, the muon mass m_μ, the neutrino mass m_ν. What is the magnitude of the muon momentum?

(b) The neutrino mass is known to be much smaller than the pion and muon masses. Indeed, one way to constrain this neutrino mass is to measure the magnitude of the momentum of the muon emitted in the decay of charged pions at rest. Compute

$$\frac{\Delta|\vec{p}|}{|\vec{p}|} \equiv \frac{|\vec{p}(m_\nu = 0)| - |\vec{p}(m_\nu)|}{|\vec{p}(m_\nu = 0)|},$$

the relative effect of a nonzero neutrino mass on the magnitude of the muon momentum. Expand your result in powers of the neutrino mass (using $m_\nu \ll m_\mu, m_\pi$), and keep only the lowest order term.

(c) In the reference frame where the pion has energy E_π, both the muon and the neutrino are emitted with a continuum of allowed energies. Compute the allowed energy range for the neutrino, assuming that $m_\nu = 0$. Make sketches of the momentum of the pion and its decay daughters when the neutrino is emitted with the largest and smallest allowed energies.

(2) **Quadrupole moment of axially symmetric charge distribution** – Recall the multipole expansion of the potential field produced by a localized charge distribution:

$$\Phi(\vec{x}) = \frac{q}{r} + \sum_i \frac{p_i x_i}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \ldots,$$

where

$$Q_{ij} \equiv \int \left(3x'_i x'_j - |\vec{x'}|^2 \delta_{ij})\rho(\vec{x'})\right) \, \mathrm{d}^3 x'$$

is the quadrupole tensor. Consider an axially symmetric ellipsoid with a uniform charge Q and a surface defined by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

(a) Show that, for the charge distribution described above, Q_{ij} is completely determined in terms of only one its elements (say, Q_{33}).

(b) Compute Q_{33} for the charge distribution described above.

(3) **Electromagnetic waves in vacuum**

(a) Starting from Maxwell's equation's in vacuum, show that the electric field and the magnetic field obey

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2},$$
$$\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}.$$

You may find the following relationship useful: $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{a}\right) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2 \vec{a}$.

(b) Let $E_x(z,0) = \alpha(z)$ and $B_y(z,0) = \beta(z)$ be the initial conditions on the field at time $t = 0$ (all other components are zero at $t = 0$). Assume that α and β are twice differentiable, continuous functions. Find $E_x(z,t)$ and $B_y(z,t)$ for all z and t.

(c) Suppose α and β vanish outside the interval $z \in [z_1, z_2]$. Find the conditions which α and β must satisfy in order that the field describes an electromagnetic wave propagating in the direction of increasing z. In particular, we will have $\vec{E}(z,t) = \vec{B}(z,t) = 0$ for all t and $z < z_1$.

(4) **Betatron** – Electrons undergoing cyclotron motion can be speeded up by increasing the magnetic field: the induced electric field will impart a tangential acceleration. Of course, it would be nice if the electron could be kept in a circular orbit during the process. Let $\vec{B}(t)$ be the value of the magnetic field at the orbit and $\phi(t)$ the magnetic flux enclosed by the orbit.

(a) Derive a relationship between $|\vec{B}(t)|$ and $\phi(t)$ such that the radius of the orbit remains constant (equal to R) as the electron accelerates. Assume that the electron starts at rest a distance R from the center, and that the initial value of the magnetic field is zero.

(b) Assuming that the magnitude of the magnetic field cannot be raised above B_{MAX}, what is the maximum energy an electron can achieve? For $R = 0.4$ m, and $B_{\mathrm{MAX}} = 0.5$ T (5000 gauss), what is the maximum electron energy? Express your answer in MeV (10^6 electron-volts).