Physics Qualifier Problems
Quantum Mechanics

September 16, 2005

Instructions:

Four problems are stated on these pages.

Solve **3** of these problems.

Please indicate which problems you want graded for credit.

Do NOT write your name on the blue book – write your *code*.

Write your answers for each problem in a separate blue book. These sheets stating the problems will be collected and discarded at the end of the exam period.

You are allowed to use the single index card you prepared in advance.
You can refer to the mathematic reference book in the room.
You are NOT allowed to use notes or books, and calculators are not allowed.

Good Luck!

(1) **One-dimensional delta-function potential**

(a) Show that the wave-function for a one-dimensional particle of mass m in the presence of a delta function potential of the form

$$V(x) = \alpha\delta(x),$$

where α may be complex, is such that the amplitude of the wave is continuous at $x = 0$, but that its slope changes (its derivative has a discontinuity at $x = 0$). Find the relation governing this slope change.

(b) Assume that an incoming wave of unit amplitude and momentum p is incident from the left on the delta-function potential described above. Calculate the probability T that this wave is transmitted and R that it is reflected.

(c) Writing $\alpha = \alpha_1 + i\alpha_2$, where $\alpha_{1,2}$ are real, use the expressions obtained in (b) to calculate the total change in the probability density associated with the incoming wave as a result of it encountering the potential. (The use of an imaginary component to a potential is sometimes used to model particle absorption, particularly in nuclear physics)

(2) **Probability conservation**

(a) Starting from the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) + V(\vec{r})\psi(\vec{x}) = i\hbar\frac{\partial\psi(\vec{x})}{\partial t},$$

where $V(\vec{x})$ is a real function, show that it can be used to obtain an equation for *probability conservation* of the form

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0.$$

Find the expression for the probability density ρ and the quantum-mechanical current $\vec{S}$.

(b) Suppose, instead, that $V(\vec{x})$ is a complex function of $\vec{x}$. What change does it make in the equation for probability conservation you derived in (a)?

(3) **Many-particle system in one-dimension**

(a) Show that an arbitrary two-particle anti-symmetric wave-function $\Psi(x_1, x_2)$ in one-dimension (two-dimensions in configuration space) may be expanded as

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}} \sum_{n,m} a_{nm} \begin{vmatrix} \psi_n(x_1) & \psi_m(x_1) \\ \psi_n(x_2) & \psi_m(x_2) \end{vmatrix},$$

where $\psi_n(x)$ form a complete orthonormal set of one-particle wave-functions. Find an expression for a_{nm} in terms of Ψ and ψ_n, and verify that it is correct.

(b) Do the same for a three-particle wave-function $\Psi(x_1, x_2, x_3)$.

(4) **Three Dimensional Scattering**

(a) The equation for the Green's function for the time independent Schrödinger equation for a particle in empty space may be written as

$$\left(\nabla^2 + k^2\right) G(\vec{r} - \vec{r'}, k^2) = \delta(\vec{r} - \vec{r'}),$$

where $E = \frac{\hbar^2 k^2}{2m}$ is the particle energy. Show that the solution to this equation is given by

$$G(\vec{r} - \vec{r'}, k^2) = -\frac{e^{ik|\vec{r} - \vec{r'}|}}{4\pi|\vec{r} - \vec{r'}|}.$$

(b) Assuming that a potential $V(\vec{r})$ is present, develop the first two terms in the Born series for the scattering amplitude for a plane-wave (wave-number $\vec{k}_i$) scattered into the solid angle $\mathrm{d}\Omega(\hat{k}_f)$, and the differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$.

(c) If the potential is given by

$$V(\vec{r}) = \begin{cases} V_0 & (0 \le r \le a), \\ 0 & (r > a), \end{cases}$$

calculate the scattering amplitude in the first Born approximation.

(d) For this same potential, calculate the s-wave scattering amplitude.

(e) Show that only the s-wave amplitude (see (d)) survives in the low energy limit. Explain the physics of this result.

(f) Find the high energy ($E \gg V_0$) limit of the cross section.