Physics Qualifier Problems
Quantum Mechanics

June 9, 2005

Instructions:

Four problems are stated on these pages.

Solve **3** of these problems.

Please indicate which problems you want graded for credit.

Do NOT write your name on the blue book – write your *code*.

Write all your answers in blue books. These sheets stating the problems will be collected and discarded at the end of the exam period.

You are allowed to use the single index card you prepared in advance.
You can refer to the mathematic reference book in the room.
You are NOT allowed to use notes or books, and calculators are not allowed.

Good Luck!

(1) Show that the elastic differential scattering cross section for a fast electron from a neutral hydrogen atom in its ground state can be written (in c.g.s. units) in the form

$$\frac{d\sigma}{d\Omega} = \left(\frac{4\,m^2\,e^4}{\hbar^4\,q^4}\right)\left[1 - f(a_0^2 q^2)\right]$$

and determine the function $f(a_0^2 q^2)$; here, $q = \Delta k$ is the momentum transfer and a_0 is the Bohr radius (neglect the effect that the electrons involved are identical particles). Discuss the physical interpretation in the limiting cases $a_0 \to 0$ and $a_0 \to \infty$.

(2) Assume you are given a many-electron state vector in the occupation number presentation of the form

$$|\Psi\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} \hat{c}^\dagger_{\mathbf{k}\uparrow} \hat{c}^\dagger_{-\mathbf{k}\downarrow}\right) |0\rangle$$

where $\hat{c}^\dagger_{\mathbf{k}\uparrow}$ creates an electron in state $+\mathbf{k}$ with spin up, $\hat{c}^\dagger_{-\mathbf{k}\downarrow}$ creates an electron in state $-\mathbf{k}$ with spin down, and $|0\rangle$ is the vacuum state.

(a) Using the appropriate commutation relations for the Fermi operators $\hat{c}^\dagger_{\mathbf{k}\alpha}$, $\hat{c}_{\mathbf{k}\alpha}$ where $\alpha = \uparrow, \downarrow$, find the condition that $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ must satisfy if the state Ψ is to be normalized such that $\langle\Psi|\Psi\rangle = 1$.

(b) Create a "Fermi sea" using the state vector in the previous problem where all states with $|\mathbf{k}| < k_F$ are filled and all states with $|\mathbf{k}| > k_F$ are empty. What form must $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ in $|\Psi\rangle$ then have?

(3) You are given three $s = 1/2$ electrons each of which has spin states $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Construct all the spin states that are eigenfunctions of the total spin operator S^2 and the z-axis projection S_z.

(4) As a model for the van der Walls interaction, we start by assuming that two isolated (but identical) atoms can be described as two uncoupled harmonic oscillators having the Hamiltonian

$$\hat{H}_0 = \frac{p_1^2}{2m} + \frac{1}{2}kx_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}kx_2^2,$$

each with a resonant frequency $\omega_0 = \sqrt{k/m}$. Now assume we have an interaction between the atoms (oscillators) of the (Coulomb) form

$$\hat{H}_1 = \frac{e^2}{R} + \frac{e^2}{R + x_1 - x_2} - \frac{e^2}{R + x_1} - \frac{e^2}{R - x_2}$$

where R is the separation between the origins of the oscillators.

To examine the behavior at large separations, expand this perturbation to leading order in $1/R$. The resulting total Hamiltonian can then be diagonalized by a canonical transformation of the form

$$x_s = \frac{1}{\sqrt{2}}(x_1 + x_2) \qquad \text{and} \qquad x_a = \frac{1}{\sqrt{2}}(x_1 - x_2)$$

with corresponding expressions for the momenta. Calculate the eigenfrequencies ω_s and ω_a corresponding to symmetric and antisymmetric modes.

Writing the zero-point energy of the system as $E = \frac{1}{2}\hbar(\omega_s + \omega_a)$, and by expanding ω_a and ω_s to the required order, show that the energy associated with the Coulomb interaction of the oscillators is given by

$$\Delta E = -\frac{A}{R^6}$$

and calculate the constant A.