Physics Qualifier Problems
Classical Mechanics and Statistical Mechanics

June 7, 2006

Instructions:

Problems 1–3 pertain to classical mechanics, while problems 4–6 pertain to statistical mechanics.

- Do **2** of the problems 1, 2 and 3.
- Do **2** of the problems 4, 5 and 6.

Thus, you will do four problems in total – two in classical mechanics, and two in statistical mechanics.

Please indicate which problems you want graded for credit.

Do NOT write your name on the blue book – write your *code*.

Write your answers for each problem in a separate blue book. These sheets stating the problems will be collected and discarded at the end of the exam period.

You are allowed to use the single index card you prepared in advance.
You can refer to the mathematic reference book in the room.
You are NOT allowed to use notes or books, and calculators are not allowed.

Good Luck!

(1) Consider a planet of mass, m, in orbit around a star of mass M. Assume further that there is a large uniform spherically symmetric distribution of dust, of density, ρ, throughout the space surrounding the star and the planet.

(a) Show that the effect of the dust is to add an additional attractive central force, $\mathbf{F} = -mC\,\mathbf{r}$, where C is a constant that depends on the gravitational constant and the density of the dust, and $\mathbf{r}$ is the radius vector from the star to the planet (both considered as points). This additional force is very small compared to the direct star-planet gravitational force.

(b) Calculate the period for a circular orbit of radius r_0 of the planet in this combined field.

(c) Calculate the period of radial oscillations for slight disturbances from this circular orbit.

(d) Show that nearly circular orbits can be approximated by a precessing ellipse and find the precession frequency. Is the precession in the same or opposite direction to the original angular velocity?

(2) A mass m_1 is constrained to move in a circle of radius a, while a second mass m_2 is constrained to move on another (concentric) circle of radius $b > a$. One can define a coordinate system where both circles are in the $x \times y$–plane, centered at the origin. The masses are connected by a spring with negligible mass and spring constant k. The spring has unstressed length $b - a$.

(a) What is the kinetic energy of the system?

(b) What is the potential energy of the system?

(c) Find the equations of motion from the Lagrangian.

(d) Calculate the angular momentum of the system about the z-axis (perpendicular to the plane defined by the two circles) and show that it is conserved. Explain why that is, even though the spring exerts forces on the masses.

(e) For small θ_1 and θ_2 (angles that define, respectively, the position of m_1 and m_2, measured from the x-axis), find the frequencies of normal–mode oscillation and the corresponding normal modes. Discuss these modes of oscillation, and construct a general state of oscillation for arbitrary $\theta_1(0)$, $\theta_2(0)$, $\dot{\theta}_1(0)$, $\dot{\theta}_2(0)$.

(3) A bar of length ℓ and negligible weight has equal point masses m at each end. The bar is made to rotate uniformly about an axis passing through its center. The bar and the axis form an angle θ $(0 < \theta < \pi/2)$.

(a) Using Euler's equation, find the components of the torque along the principal axes of the bar.

(b) Find the components of the torque and the angular momentum vector in an inertial reference frame with origin at the center of the bar.

(c) Show that the results of (b) are consistent with those of (a). Explain why.

(4) Consider an ideal gas made up of N identical diatomic molecules, each composed of two dissimilar atoms. Let the gas be in thermal equilibrium at temperature T, and let the moment of inertia of each molecule be I. A quantum mechanical analysis of one such molecule shows that the angular momentum of the Jth level equals $\sqrt{J(J+1)}\hbar$, and its degeneracy is $2J+1$. Find the rms rotational velocity in the high-temperature limit, using two different methods:

(i) Use the equipartition theorem;

(ii) Start with the exact quantum mechanical partition function, but approximate the sum as an integral.

Show that your answers agree, and state how high the temperature needs to be in order for your expressions for the rms rotational velocity to be valid.

(5) This problem deals with the distribution of air in the Earth's atmosphere. Ignore the Earth's curvature, and assume that the gravitational acceleration g is independent of height z above the surface. For simplicity, treat the atmosphere as a monatomic ideal gas composed of particles of mass m.

  (a) Assuming a constant temperature T_0, find $P(z)$ and $\rho(z)$, the pressure and mass density as a function of height. Express your answer in terms of g, m, and T_0.

  (b) Now let the temperature be T_s at the surface, but let it decrease with increasing z (as is true for the troposphere – the bottom 10 to 15 km of Earth's atmosphere). Under adiabatic conditions, if the temperature gradient at any height z exceeds a certain critical value, then convection will occur. Thus, temperature gradients larger than this critical value are often suppressed. Find $P(z)$ and $\rho(z)$ under the assumption that the temperature gradient at every height is equal to the critical value. Express your answer in terms of g, T_s, and m.

(6) Consider a solid made up of weakly interacting magnetic atoms, with number density n and temperature T. In this problem, you are asked to treat this situation classically. In other words, let the magnetic dipoles (each with magnetic moment μ) be free to point in any direction, regardless of the direction of the uniform external field, taken to have magnitude B. Show that the magnetization per unit volume is given by the following expression:

$$M = n\mu \left[\coth\left(\frac{\mu B}{k_B T}\right) - \left(\frac{k_B T}{\mu B}\right)\right],$$

where k_B is the Boltzmann constant.