Physics Qualifier Problems
Quantum Mechanics

June 9, 2006

Instructions:

Four problems are stated on these pages.

Solve **3** of these problems.

Please indicate which problems you want graded for credit.

Do NOT write your name on the blue book – write your *code*.

Write your answers for each problem in a separate blue book. These sheets stating the problems will be collected and discarded at the end of the exam period.

You are allowed to use the single index card you prepared in advance.
You can refer to the mathematic reference book in the room.
You are NOT allowed to use notes or books, and calculators are not allowed.

Good Luck!

(1) A beam of neutrons incident along the z-axis scatters from a hydrogen molecule oriented such that the line joining the two nuclei lies along the z-direction. Assume you can model the interaction of the neutron with the two protons by the potential

$$V(r) = \alpha \left[\delta(x)\delta(y)\delta(z-a) + \delta(x)\delta(y)\delta(z+a) \right],$$

where $2a$ is the separation between the nuclei. Calculate the scattering amplitude and the differential scattering cross section using the Born approximation.

(2) An electron is bound to the planar surface of liquid helium (located at $z = 0$) by a potential involving a combination of its own dielectric image charge and a repulsive barrier prohibiting entry of the electron into the liquid that we can write as

$$V(z) = \begin{cases} -\alpha \frac{e^2}{2z}; & z > 0, \\ +\infty; & z < 0, \end{cases}$$

where $\alpha = (\varepsilon_{\text{He}} - 1) / (\varepsilon_{\text{He}} + 1)$, with ε_{He} being the dielectric constant of helium.

(a) Write the Hamiltonian for this problem in Cartesian coordinates.

(b) Carry out a separation of variables by writing the total wave function as the product $\Psi(x, y, z) = F(x, y)Z(z)$ and writing the energy as $E = E_{\parallel} + E_{\perp}$ where the latter is associated with separation constants involving the motion parallel and perpendicular to the surface, respectively. What is the form of the function $F(x, y)$?

(c) For the *lowest* energy state find the expression for the binding energy as well as the normalized function $Z(z)$ (which has no nodes). (You may find it helpful to try a form that vanishes rapidly for large (positive) z and satisfies the appropriate boundary condition at $z = 0$.)

(3) Consider a hydrogen atom in its ground state between two parallel metallic plates of a capacitor. An electric field is turned on at time $t = 0$, and then dies away according to

$$E(t) = \begin{cases} 0; & t < 0, \\ E_0 e^{-t/\tau}; & t > 0, \end{cases}$$

where τ is the field's lifetime. The interaction Hamiltonian for the atom in the uniform electric field is given by $\hat{H} = -e\vec{r} \cdot \vec{E}(t)$, where e is the charge of the electron and $\vec{r}$ is the position of the electron relative to the proton.

(a) Starting from the time-dependent Schrödinger equation write down the time dependent state vectors for the ground state and the excited state with principle quantum number $n = 2$ *in the absence of the electric field.*

(b) Use time-dependent perturbation theory to calculate the correction to the ground state for times $t > 0$ to first-order in the electric field. Hint: you may find it useful to transform to the interaction picture.

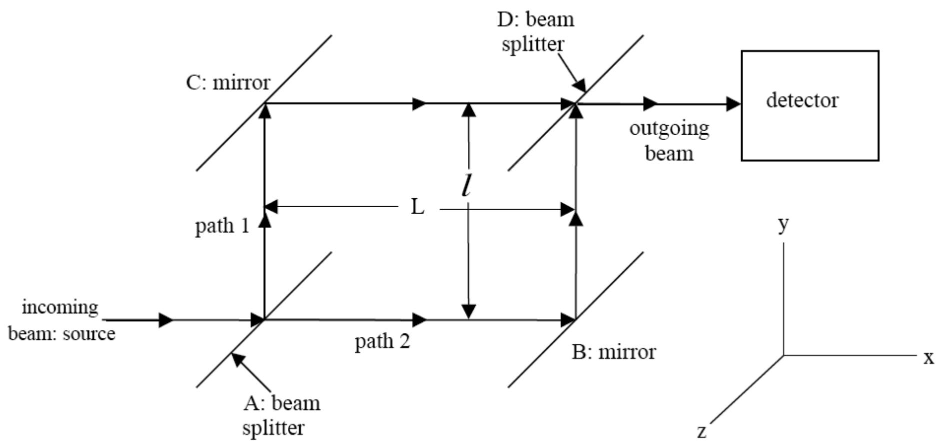
(c) Calculate the probabilities for the atom to be found in the various first excited states, i.e., the $n = 2$ levels, to leading order in E_0. Be sure to specify the orbital quantization axis you choose and the quantum numbers for the excited states.

Note the non-relativistic hydrogenic wavefunctions ψ_{nlm} are given by

$$\begin{aligned}
\psi_{100} &= \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \\
\psi_{200} &= \frac{1}{4\sqrt{2\pi a^3}} \left(2 - \frac{r}{a}\right) e^{-r/2a}, \\
\psi_{210} &= \frac{1}{4\sqrt{2\pi a^3}} \left(\frac{r}{a}\right) e^{-r/2a} \cos\theta, \\
\psi_{21\pm1} &= \frac{1}{8\sqrt{\pi a^3}} \left(\frac{r}{a}\right) e^{-r/2a} \sin\theta e^{\pm i\phi}, \\
\psi_{300} &= \frac{1}{81\sqrt{3\pi a^3}} \left(27 - 18\frac{r}{a} + 2\left(\frac{r}{a}\right)^2\right) e^{-r/3a}, \\
\psi_{310} &= \frac{\sqrt{2}}{81\sqrt{\pi a^3}} \left(6 - \frac{r}{a}\right) \frac{r}{a} e^{-r/3a} \cos\theta, \\
\psi_{31\pm1} &= \frac{1}{81\sqrt{\pi a^3}} \left(6 - \frac{r}{a}\right) \frac{r}{a} e^{-r/3a} \sin\theta e^{\pm i\phi}, \\
\psi_{320} &= \frac{1}{81\sqrt{6\pi a^3}} \left(\frac{r}{a}\right)^2 e^{-r/3a} \left(3\cos^2\theta - 1\right), \\
\psi_{32\pm1} &= \frac{1}{81\sqrt{\pi a^3}} \left(\frac{r}{a}\right)^2 e^{-r/3a} \sin\theta\cos\theta e^{\pm i\phi}, \\
\psi_{32\pm2} &= \frac{1}{162\sqrt{\pi a^3}} \left(\frac{r}{a}\right)^2 e^{-r/3a} \sin^2\theta e^{\pm 2i\phi},
\end{aligned}$$

where a is the Bohr radius.

(4) Consider a nearly monoenergetic beam of neutrons initially prepared as a wavepacket, $\psi(\vec{r})$, with a well defined momentum $\vec{p}_0 = p_0\hat{x}$, i.e. a momentum spread $\Delta p_x \ll p_0$. The neutron beam is split into two coherent beams by Bragg reflection from a silicon crystal at position A and reflected by Bragg mirrors at positions B and C, then recombined at position D in front of the detector as indicated in the diagram. One beam travels along path ABD, while the other beam travels along path ACD. If the plane of the interferometer is perpendicular to the gravitational acceleration, $\vec{g} = -g\hat{z}$, then neutrons propagating along either of the two paths pass through the same gravitational potential. However, if the plane of the interferometer is inclined by and angle δ about the axis AB then the gravitational potential is different for the neutrons propagating along path ABD and path ACD.



(a) Write down the Hamiltonian that determines the time evolution of a neutron in a gravitational field of uniform acceleration, $\vec{g}$.

(b) Calculate the phase difference of the wavepackets which arrive at the detector from the two paths ABD and ACD when the interferometer is tilted by angle δ. Express the difference in phase in terms of the neutron mass, M_n, the acceleration due to gravity, g, the dimensions of the interferometer, L and l, the de Broglie wavelength of the neutrons, λ, the tilt angle δ and relevant fundamental constants.

(c) In an actual version of this experiment the number of neutrons arriving at the detector is represented in the figure below. If the dimensions of the interferometer are $L = 10$ cm and $l = 1$ cm, calculate the wavelength of the incident neutrons.