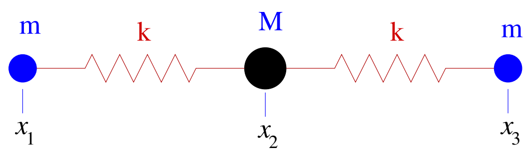
Northwestern University Physics Qualifying Exam
Wednesday, September 13, 2006

Classical and Statistical Mechanics

This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the exam for a total of four problems.

Do each problem in a separate blue book and write your ID number (not your name) and the problem number on each book. If your solution uses more than one book, label each book with 1 out of 2, 2 out of 2...

Classical Mechanics - do 2 out of 3 problems



1. The linear one dimensional oscillations of a symmetrical triatomic molecule can be modeled by the configuration of a mass M connected by springs of constant force constant k, to two equal masses m, as shown. For the purposes of this model, assume that the masses are constrained to lie on a straight line. Let x_1, x_2 and x_3 represent the displacements of the masses from their respective equilibrium positions.

   (a) What is the kinetic energy T of the system.

   (b) What is the potential energy V of the system.

   (c) From the Lagrangian $\mathcal{L} = T - V$ find the equations of motion for the system.

   (d) Calculate the frequencies of the normal modes of the system.

   (e) Find the corresponding amplitudes functions of the normal modes and give the physical interpretation of each mode.

2. A railroad flatcar of mass M can roll without friction along a straight horizontal track. N men, each of mass m are initially standing on the car which is at rest.

   (a) the N men run to one end of the car in unison so that their speed relative to the car is V_r just before they jump off (all at the same time). Calculate the velocity of the car after the men have jumped off.

   (b) The N men run off the car, one after the other, (only one man running at a time) each reaching a speed of V_r relative to the car just before jumping off. Find an expression for the final velocity of the car.

   (c) In which case, (a) or (b), does the car attain the greater velocity?

3. A point mass m moves in the 3-dimensional attractive potential $-k/r$ with total energy E and angular momentum l.

   (a) Describe the possible orbits for E positive, 0 and negative. Be as quantitative as possible, providing the general formula for the orbits and defining the parameters which characterize those orbits.

   (b) For circular and parabolic orbits having the same angular momentum $l \neq 0$ show that the perihelion distance of the parabolic orbit, a, is $1/2$ of the radius, R, of the circular orbit.

   (c) At the same point as in part (b) show that the speed of the parabolic orbit is twice the speed of the circular orbit.

   (d) At the point where the orbits intersect (at radius R) show that the speed of the parabolic orbit is $\sqrt{2}$ of the speed of the circular orbit.

Statistical Mechanics - do 2 out of 3 problems

1. An aqueous solution at room temperature T contains a very weak concentration of magnetic atoms, each of which has a net spin $\frac{1}{2}$ and a magnetic moment μ. The solution is placed in a static external magnetic field pointing along the z direction. The magnitude of this field is inhomogeneous over the volume of the solution. Specifically, $B = B(z)$ is a monotonic increasing function of z, assuming a value B_1 at the bottom of the solution where $z = z_1$ and a larger value B_2 at the top of the solution where $z = z_2$. Let $n(z)dz$ denote the total mean number of magnetic atoms (irrespective of spin orientation) located between z and $z + dz$. Under the assumption that $\mu B \ll kT$, show that to a good approximation $n(z_2)/n(z_1)$ will be a linear function of ($B_2{}^2$ - $B_1{}^2$).

2. Consider a single atom of mass m in a 3-d box of dimensions L_x, L_y, and L_z. The quantum mechanical states have energies given by

$$E(n_x, n_y, n_z) = (h^2/8m)[(n_x/L_x)^2 + (n_x/L_x)^2 + (n_x/L_x)^2]$$

   (a) Starting from this result, show that the partition function for temperature T is given by $Z_1 = (V/v_q)$, where V is the volume of the box and

$$v_q = (\frac{h^2}{2\pi mkT})^{3/2}$$

   Work in the high temperature limit.

   (b) Using your result from part a, find the partition function and the chemical potential for an ideal gas consisting of N indistinguishable atoms, as a function of V, T, N, and m.

   (c) Next consider a gas of N' weakly interacting atoms, adsorbed on a surface of area A on which they are free to move. This situation can be treated as a two-dimensional ideal gas on the surface. The energy of an adsorbed atom then equals the translational kinetic energy minus the binding energy ϵ that holds each atom to the surface. Calculate the partition function of this adsorbed gas, again working in the high temperature limit.

   (d) Using your result for part c, calculate the chemical potential μ' for the adsorbed gas.

   (e) Consider a situation in which atoms adsorbed on a surface are in diffusive and thermal equilibrium with a reservoir of identical atoms in the surrounding three-dimensional gas. Derive an expression for the mean number n' of atoms adsorbed per unit area of the surface as a function of the mean pressure p of the surrounding gas, the binding energy ϵ, the temperature T, and the atomic mass m.

3. Einstein's quantization of electromagnetic radiation assumes that radiation is composed of photons each with energy $\varepsilon_{\mathbf{k}} = \hbar\omega_{\mathbf{k}} = \hbar c|\mathbf{k}|$, where $\omega_{\mathbf{k}}$ is the frequency of a mode with wavevector $\mathbf{k}$, $\hbar = h/2\pi$, h is Planck's constant and c is the speed of light. Thus, the energy of $n_{\mathbf{k}}$ photons is simply $\varepsilon_n(\mathbf{k}) = n_{\mathbf{k}}\,\varepsilon_{\mathbf{k}}$ where $n_{\mathbf{k}} \in \{0, 1, 2, \ldots\}$, i.e. photons are Bosons.

   (a) Evaluate the mean number of photons for a mode of wavenumber $\mathbf{k}$, in equilibrium with a heat bath at temperature T,

   $$\bar{n}_{\mathbf{k}} = \frac{1}{\mathcal{Z}} \sum_{n_{\mathbf{k}}=0}^{\infty} n_{\mathbf{k}}\, e^{-\varepsilon_n(\mathbf{k})/k_B T}\,, \tag{1}$$

   where k_B is Boltzmann's constant. You need only do the sum for the partition function, $\mathcal{Z} = \sum_{n_{\mathbf{k}}=0}^{\infty} e^{-\varepsilon_n(\mathbf{k})/k_B T}$, in order to compute $\bar{n}_{\mathbf{k}}$.

   (b) The Planck distribution for the *spectral energy density* of radiation in equilibrium at temperature T is given by,

   $$\rho_\omega = \frac{\hbar\omega^3}{\pi^2\, c^3}\, \frac{1}{e^{\hbar\omega/k_B T} - 1}\,, \tag{2}$$

   where ω is the angular frequency of the radiation. The mean energy density is then $\bar{\varepsilon}(T) = \int_0^\infty d\omega\, \rho_\omega$. Derive the Planck distribution by summing over all modes $\mathbf{k}$ and photon polarizations. Express the distribution of radiation in terms of a wavelength distribution, i.e. obtain ρ_λ where $\bar{\varepsilon}(T) = \int_0^\infty d\lambda\, \rho_\lambda$.

   (c) Derive Wien's law for the wavelength corresponding to the peak in the spectral density, i.e. show that

   $$\lambda_{\text{peak}} = \frac{\Lambda}{T}\,, \tag{3}$$

   and evaluate the coefficient Λ in units of $[\mathring{A} \cdot K]$.

   (d) The total radiant energy density at temperature T scales as

   $$\bar{\varepsilon}(T) = \sigma\, T^p\,. \tag{4}$$

   Determine the power p, and evaluate the Stefan-Boltzmann constant, σ, in units of $J \cdot m^{-3} \cdot K^{-p}$.