1. The $1s$ levels of positronium (*i.e.* the Hydrogenic bound state of an electron and a positron) in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ are approximately described by the Hamiltonian

$$\mathcal{H} = E_{1s} + A\mathbf{S}_e \cdot \mathbf{S}_p + \frac{|e|}{mc}(S_{ez} - S_{pz})B\,, \qquad (5)$$

where E_{1s} is the ground state energy in the non-relativistic approximation, e and m are the charge and mass of the electron, c is the speed of light and $\mathbf{S}_{p(e)}$ is the spin operator for the positron (electron).

(a) Calculate the ionization energy of positronium neglecting the spin-dependent terms. Express your answer in terms of (i) the rest mass energy of the electron (mc^2) and the fine structure constant (α), (ii) in terms of the Rydberg (R_{H}) and (iii) in terms of eV.

(b) Calculate the energies of the $1s$ levels for $B = 0$ using the Hamiltonian in Eq. (Hyperfine-Levels). Calculate the wavelength of a photon that is produced in the radiative decay of the $1\,^3s \to 1\,^1s$ transition for $B = 0$ assuming the coefficient A originates from the interaction between the magentic dipole moment of the electron ($\boldsymbol{\mu}_e$) and positron ($\boldsymbol{\mu}_p$),

$$\widehat{H}_{\text{dipole}} = -3\frac{(\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\mu}}_e)(\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\mu}}_p)}{\hat{r}^5} + \frac{\hat{\boldsymbol{\mu}}_e \cdot \hat{\boldsymbol{\mu}}_p}{\hat{r}^3} - \frac{8\pi}{3}\,\hat{\boldsymbol{\mu}}_e \cdot \hat{\boldsymbol{\mu}}_p\,\delta^{(3)}(\hat{\mathbf{r}})\,. \qquad (6)$$

(c) Using the basis which diagonalizes the spin Hamiltonian for $B = 0$ obtain the energy levels and eigenvectors of the $1s$ levels for $B \neq 0$. Sketch the evolution of the energy levels as a function of magnetic field and identify the spin states in both the high- and low-field limits.

2. A particle of mass m and charge e is confined by a harmonic potential in one dimension. The restoring force and mass define the natural frequency, ω, of this oscillator, and the Hamiltonian governing its dynamics is:

$$\widehat{H}_0 = \frac{1}{2m}\,\widehat{p}^2 + \frac{1}{2}m\omega^2\,\widehat{x}^2\,. \tag{7}$$

where $\widehat{p}$ is the momentum operator and $\widehat{x}$ is the canonically conjugate position operator.

(a) The Hamiltonian can be put into dimensionless form,

$$\widehat{H}_0 = \frac{1}{2}\epsilon_0\left(\widehat{\widehat{p}}^2 + \widehat{\widehat{x}}^2\right)\,, \tag{8}$$

where $\widehat{\widehat{x}} = \widehat{x}/\ell$ and $\widehat{\widehat{p}} = \widehat{p}/(\hbar/\ell)$ are dimensionless operators obtained by scaling length in units of ℓ, momentum in units of $\hbar/\ell$, and energy in units of ϵ_0. Determine ϵ_0 and ℓ.

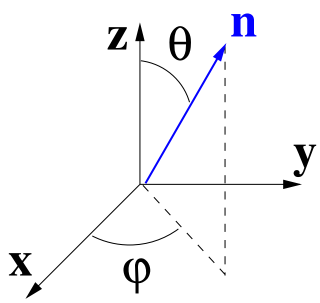
(b) Show that $a^\dagger = (\widehat{\widehat{x}} - i\widehat{\widehat{p}})/\sqrt{2}$ and $a = (\widehat{\widehat{x}} + i\widehat{\widehat{p}})/\sqrt{2}$ are the raising and lowering operators for the quantum oscillator, and that $\widehat{n} = a^\dagger a$ is the operator corresponding to the number of quanta of excitation of the oscillator. Express the Hamiltonian in terms of these operators.

(c) A static electric field, $\mathcal{E}$, is applied along the x-axis. The Hamiltonian now includes the coupling of the electric dipole moment, $\widehat{\mathcal{D}} = e\widehat{x}$, to the field,

$$\widehat{H} = \widehat{H}_0 - e\mathcal{E}\,\widehat{x}\,. \tag{9}$$

For small electric fields use perturbation theory to calculate the change in the ground state energy to second order in $\mathcal{E}$. Use this result to calculate the polarizability of the oscillator, i.e. $\alpha = d\mathcal{D}/d\mathcal{E}$ where $\mathcal{D}$ is the mean dipole moment ("polarization") of the oscillator in its ground state.

(d) The oscillator in a static electric field can be solved exactly by an appropriate shift of the ground state energy and the center of force acting on the particle. Calculate the exact ground state energy and polarizability and compare with the results from perturbation theory. What is special about the result for the polarizability?



3. The spin angular momentum of a neutron is described by an operator

$$\widehat{\mathbf{S}} = \frac{\hbar}{2}\boldsymbol{\sigma}\,. \tag{10}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices. In the basis of spin states corresponding to neutrons polarized along $\pm\hat{\mathbf{z}}$ in the laboratory coordinate system

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{11}$$

The neutron interacts with a magnetic field, $\mathbf{B}$, through its magnetic moment. The Hamiltonian is

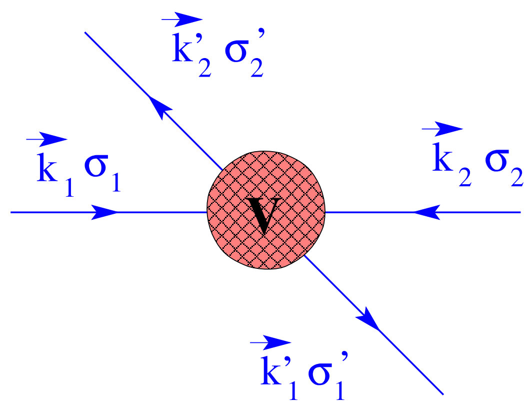
$$\widehat{H} = -\gamma\widehat{\mathbf{S}}\cdot\mathbf{B}\,. \tag{12}$$

where γ is the neutron gyromagnetic ratio.

(a) A Stern-Gerlach apparatus is used to measure the spin projection of neutrons along an axis $\hat{\mathbf{n}}$ oriented as shown in the laboratory coordinate system. Find the eigenvalues and eigenvectors of the spin projected along $\mathbf{n}$.

(b) Consider neutrons with spin perfectly polarized along the $+\mathbf{n}$ direction. What is the orientation of $\mathbf{n}$ and the corresponding state vector $|\,\psi\,\rangle$ for the neutrons if measurements of $\widehat{S}_z$ yield a mean value $\langle\,\psi\,|\widehat{S}_z|\,\psi\,\rangle = 0$? Show that it is not possible construct a state $|\,\psi\,\rangle$ in which $\langle\,\psi\,|\widehat{S}_x|\,\psi\,\rangle = \langle\,\psi\,|\widehat{S}_y|\,\psi\,\rangle = \langle\,\psi\,|\widehat{S}_z|\,\psi\,\rangle = 0$.

(c) A beam of un-polarized neutrons is passed through a Stern-Gerlach apparatus in order to produce a perfectly polarized beam with spins align along $+\hat{\mathbf{z}}$. A uniform magnetic field $\mathbf{B} = B\hat{\mathbf{y}}$ is turned on at time 0. After a time t the beam is analyzed by a second Stern-Gerlach apparatus which only passes those with spins polarized along the $+\hat{\mathbf{x}}$ axis. What fraction of the original particles pass through the second Stern-Gerlach apparatus?



4. Two non-relativistic spin-1/2 particles with unequal masses, m_1 and m_2, scatter off one another. The Hamiltonian governing their dynamics is,

$$\widehat{H} = \frac{|\widehat{\mathbf{p}}_1|^2}{2m_1} + \frac{|\widehat{\mathbf{p}}_2|^2}{2m_2} + \alpha \frac{1}{|\widehat{\mathbf{r}}_1 - \widehat{\mathbf{r}}_2|} e^{-|\widehat{\mathbf{r}}_1 - \widehat{\mathbf{r}}_2|/a}\, \widehat{\mathbf{s}}_1 \cdot \widehat{\mathbf{s}}_2 \,, \tag{13}$$

where $\widehat{\mathbf{p}}_1$ ($\widehat{\mathbf{p}}_2$) is the momentum operator for particle 1 (2), $\widehat{\mathbf{r}}_1$ and $\widehat{\mathbf{r}}_2$ are the canonically conjugate position operators, and $\widehat{\mathbf{s}}_1$ and $\widehat{\mathbf{s}}_2$ are the corresponding spin operators. The figure depicts the initial ($|\, \mathbf{k}_1\sigma_1, \mathbf{k}_2\sigma_2 \,\rangle$) and final ($|\, \mathbf{k}'_1\sigma'_1, \mathbf{k}'_2\sigma'_2 \,\rangle$) states labeled by the momenta and spin projections for particles 1 and 2.

(a) For elastic scattering which of the following transitions are forbidden?

i. $|\, +\mathbf{k}/2 \uparrow, -\mathbf{k}/2 \uparrow\rangle \longrightarrow |\, +\mathbf{k}'/2 \uparrow, -\mathbf{k}'/2 \uparrow\rangle$

ii. $|\, +\mathbf{k}/2 \uparrow, -\mathbf{k}/2 \uparrow\rangle \longrightarrow |\, -\mathbf{k}/2 \uparrow, +\mathbf{k}/2 \uparrow\rangle$

iii. $|\, +\mathbf{k}/2 \uparrow, -\mathbf{k}/2 \uparrow\rangle \longrightarrow |\, +\mathbf{k}'/2 \downarrow, -\mathbf{k}'/2 \downarrow\rangle$

iv. $|\, +\mathbf{k}/2 \uparrow, -\mathbf{k}/2 \uparrow\rangle \longrightarrow |\, -\mathbf{k}'/2 \uparrow, +\mathbf{k}'/2 \downarrow\rangle$

v. $|\, +\mathbf{k}/2 \uparrow, -\mathbf{k}/2 \downarrow\rangle \longrightarrow |\, +\mathbf{k}'/2 \uparrow, -\mathbf{k}'/2 \downarrow\rangle$

vi. $|\, +\mathbf{k}/2 \uparrow, -\mathbf{k}/2 \downarrow\rangle \longrightarrow |\, +\mathbf{k}'/2 \downarrow, +\mathbf{k}'/2 \uparrow\rangle$

vii. $|\, +\mathbf{k}/2 \uparrow, -\mathbf{k}/2 \downarrow\rangle \longrightarrow |\, +\mathbf{k}'/2 \downarrow, -\mathbf{k}'/2 \uparrow\rangle$

For those transitions that are forbidden what symmetry is at work?

(b) Calculate the ratio of the total scattering cross-sections for initial states with total spin $S = 1$ and total spin $S = 0$ in the first Born approximation. If you don't remember the formula for the first Born approximation, then start from Fermi's Golden rule and derive it.

(c) Calculate the differential cross-section for initial states with total spin $S = 0$ in the first Born approximation.