Northwestern University Physics Qualifying Examination

Friday, September 21, 2007

Classical and Statistical Mechanics

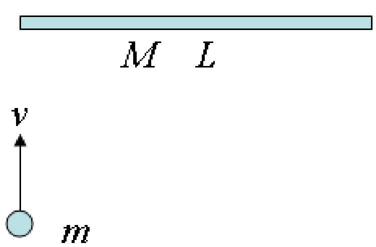
This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the examination for a total of four solved problems.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

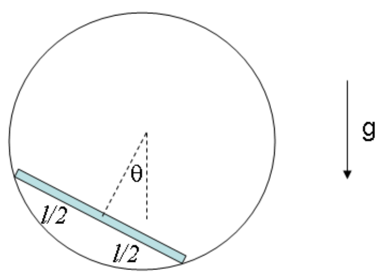
Classical Mechanics - do 2 out of 3 problems

1. Satellite motion:
A satellite of mass m moves in a circular orbit around the Earth (of mass M) at radius R and speed v. Due to the firing of a rocket, the satellite suddenly acquires an additional outward radial velocity v. Assume the mass of the satellite is unchanged.
(a) Calculate the change in the total energy of the satellite. Express your answer in terms of m, M, R and G.
(b) Calculate the ratio of the final angular momentum to the initial angular momentum.
(c) Describe the motion of the satellite after the explosion.

2. Elastic collision:
A ball of mass m moves with speed v on a frictionless horizontal plane. It collides elastically with one end of a uniform bar of length L and mass M. The velocity v is perpendicular to the bar, which is initially at rest on the plane. Describe the motion of the ball and the bar after the collision. The moment of inertia of the bar about its center is $M\,L^2/12$, and the moment of inertia of the bar about one of its end is $M\,L^2/3$.



3. Small-amplitude oscillation:
A uniform bar of length l and mass m slides without friction inside a circular track of radius R. The ends of the bar are constrained to be on the track. $l = \sqrt{2}R$.
(a) Find the Lagrangian of the bar using the angle θ as the coordinate.
(b) Derive the equation of motion for the bar.
(c) What is the frequency of small oscillation around the equilibrium position?



Statistical Mechanics - do 2 out of 3 problems

1. Thermal Fluctuations of a Pendulum: A small pendulum of fixed length ℓ hangs in a gravitational field that exerts a force f in the $-\hat{\mathbf{z}}$ direction. The mass of the pendulum is concentrated at its end. We isolate the pendulum from all other external forces, and leave it alone so that it reaches thermal equilibrium at temperature T. You may consider its motions to be governed by canonical classical statistical mechanics.

(a) Briefly describe the motion of the pendulum in thermal equilibrium.

(b) What is the potential energy as a function of its small displacement in the x-y plane, to order x^2 and y^2?

(c) Find a formula relating the mean-square-fluctuations of the in-plane position, $\langle x^2 + y^2 \rangle$, to the force f. You may consider the fluctuations to be small enough to use only the harmonic approximation of (b).

(d) What is the probability distribution of the displacement of the pendulum from its mechanical equilibrium position, for small displacements in the x-y plane?

2. Velocity Distribution of an Ideal Gas: Consider an ideal gas in a box, in equilibrium at temperature T. The particles each have kinetic energy $m\mathbf{v}^2/2$ and are spinless point particles. They are at sufficiently low density that their quantum statistics are unimportant.

The box is made of a thin but impermeable material, and is surrounded by vacuum.

(a) Find the normalized velocity distribution for the particles inside the sealed box, $P(v_x, v_y, v_z)$, and calculate their average speed, $\langle|\mathbf{v}|\rangle$.

Now, suppose that a small hole of cross-sectional area a is made in the box. However, the hole diameter is much larger than the thickness of the material that the box is made of. Particles will start to escape from the box. The normal to the surface where the hole is made is in the $+z$ direction.

(b) Find the number of particles dN that escape with velocity $\mathbf{v}$, per time interval dt and per velocity element d^3v, just after the hole is made.
(Hint: your result should be a function of the magnitude of velocity and the angle of velocity relative to the $+z$ direction)

(c) What is the total rate of particle escape (dN/dt) just after the hole is made? Your result should be expressed in terms of k_BT, m, a, N and V plus numerical factors.

(d) What is the average energy per escaping particle?

3. Extensible molecule: Consider a molecule which is a linear series of N identical units, each of which can be in one of two states, say 0 and 1.

In state 0, the length of a unit is ℓ_0, while in state 1, the length of a unit is ℓ_1. Therefore, if n units are in state 0 and $N - n$ units are in state 1, the molecule as a whole will have length $L = N\ell_0 + (\ell_1 - \ell_0)n$.

The energy of each unit in state 1 is ϵ larger than the energy in state 0, *i.e.* the energy is $E = n\epsilon$ where n is the number of units in state 1 ($0 \leq n \leq N$).

(a) Find the entropy of the molecule for fixed energy (fixed n).

(b) At constant temperature, find the canonical partition function.

(c) Calculate the average total length of the molecule at constant temperature.

(d) Calculate the mean-square fluctuation in total length of the molecule at constant temperature.

Constants

$k_B = 1.381 \times 10^{-23}$ J/K $\quad h = 6.626 \times 10^{-34}$ J·sec $\quad c = 2.998 \times 10^8$ m/sec
$m_p = 1.673 \times 10^{-27}$ kg $\quad m_n = 1.675 \times 10^{-27}$ kg $\quad m_e = 9.109 \times 10^{-31}$ kg
$e = 1.602 \times 10^{-19}$ C $\quad G = 6.67 \times 10^{-11}$ N m^2/kg^2

At room temperature $T = 300$ K and $k_B T = 4.1 \times 10^{-21}$ J

Integrals

$$\int_{-\infty}^{\infty} dx \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx \ x^n \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n$$

for $n = 2, 4, 6 \cdots$

Series

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$1 + x + x^2 + \cdots x^{P-1} = \frac{1 - x^P}{1 - x} \qquad \text{(geometric series)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \text{Zeta function} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \qquad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056 \cdots$$

Hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx} \sinh x = \cosh x \qquad \frac{d}{dx} \cosh x = \sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

Stirling's approximation for log of factorial

$$\ln n! = n \ln n - n + \mathcal{O}(1)$$

Combinations

$$C_n^N = \frac{N!}{n!(N-n)!}$$

Classical ideal gas partition function

$$Z(N,T,V) = \frac{1}{N!}\left(\frac{1}{h^3}\int_V d^3r \int d^3p \;\; \exp\left[-\frac{\mathbf{p}^2}{2mk_BT}\right]\right)^N$$

$$= \frac{V^N}{N!}\left(\frac{2\pi m k_B T}{h^2}\right)^{3N/2}$$