Northwestern University Physics Qualifying Examination

Thursday, September 20, 2007

Electricity and Magnetism

Solve 3 out of 4 problems

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

2007 Fall Qualifier Exam
Electricity and Magnetism

Note: you should do only three of the following four problems

Problem 1

(a) Using Maxwell's equations in free space, show that in the presence of externally driven sources (currents and potentials) the vector and scalar potentials satisfy the equations

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r},t) = -\mu_0 \mathbf{j}(\mathbf{r},t) \tag{1}$$

and

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{r},t) = -\frac{\rho(\mathbf{r},t)}{\varepsilon_0} \tag{2}$$

provided that the potentials satisfy what is called the Lorentz gauge condition:

$$\nabla \cdot \mathbf{A}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial \phi(\mathbf{r},t)}{\partial t} = 0\,. \tag{3}$$

(b) Assume that we are dealing with a single frequency, ω, such that all quantities are proportional to $e^{-i\omega t}$; i.e., $\mathbf{A}(\mathbf{r},t) = \mathbf{A}(\mathbf{r})e^{-i\omega t}$ and $\phi(\mathbf{r},t) = \phi(\mathbf{r})e^{-i\omega t}$. Show that if we write $\mathbf{j}(\mathbf{r},t) = \mathbf{j}(\mathbf{r})e^{-i\omega t}$, then $\mathbf{A}(\mathbf{r})$ in free space is given by:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\, \mathbf{j}(\mathbf{r}')\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{4}$$

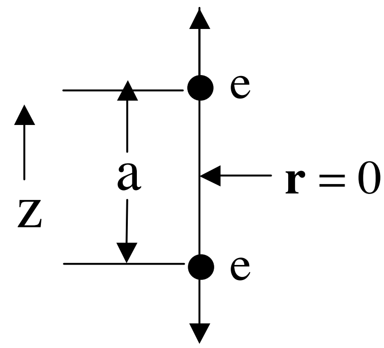
where $k = \omega / c$ with $k = 2\pi / \lambda$ and λ the wavelength.

(c) In the so-called dipole limit where the size of the source region is much less than a wavelength of light, we can write

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 e^{ikr}}{4\pi r}\int d^3r'\, \mathbf{j}(\mathbf{r}')\ . \tag{5}$$

Now suppose the current density arises from a *single* oscillating electron of charge e with a velocity having amplitude $\mathbf{v} = v_0\,\hat{\mathbf{z}}$ *about the origin* such that the current density is given by $\mathbf{j}(\mathbf{r}') = e\delta^{(3)}(\mathbf{r}')\hat{\mathbf{z}}v_0$. Calculate the resulting electric and magnetic fields **E** and **H** at distances $r >> \lambda$. (Hint: first calculate the magnetic field using the usual relation between **B** and **A**, and then obtain the electric field from Maxwell's fourth equation in the region where $\mathbf{j} = 0$.)

(d) Now assume there are *two electrons*, each located *away from the origin*, one at $z = +a/2$ and the other at $z = -a/2$ (see figure) with the distance $a \ll \lambda$ where λ is the wavelength; furthermore assume these two electrons are oscillating 180° out of phase. Obtain an expressions for **A** as well as the associated electric and magnetic fields **E** and **H** for the case where $r \gg \lambda$.



Problem 2

(a) Suppose we envision a solid as consisting of electrons of mass m and charge e connected to bound nuclei (ions) by springs with an isotropic spring constant k. We can write the Newtonian equation of motion for the electron as

$$m\ddot{\mathbf{r}} + \frac{m}{\tau}\dot{\mathbf{r}} + k\mathbf{r} = e\mathbf{E}(t).$$

We will assume an oscillatory internal electric field of the form $\mathbf{E}(t) = \mathbf{E}_0 e^{-i\omega t}$ is present. Now in the theory of dielectrics the polarization vector, **P**, is defined as the dipole moment per unit volume. If the dipole moment of each atom in the solid is written as $\mathbf{p} = e\mathbf{r}$, and the number atoms per unit volume is n, obtain expressions for the polarization vector, **P**(t), the displacement vector, **D**(t), and dielectric constant, $\varepsilon(\omega)$, of the solid. Write the latter in terms of the resonant frequency, $\omega_0 = \sqrt{\frac{k}{m}}$, and a quantity $\omega_p = \sqrt{\frac{ne^2}{\varepsilon_0 m}}$, which has the units of frequency (and is called the plasma frequency).

(b) As a model of a good metal (where the electrons are free) set the spring constant $k = 0$ and $\tau = \infty$ in the expression for $\varepsilon(\omega)$ obtained in part (a). Assume you are given such a metal that occupies the half space $z < 0$. At what frequency can one sustain an electric field oscillating in the z-direction *inside* the metal while having it vanish *outside* the metal.

Problem 3

The inductance of a coil, L, which depends on many factors, is related to the current, i, and flux, Φ, that pass through it by the relation $\Phi = Li$. When there are two such coils, where some of flux from one passes through the other, the flux Φ_1 passing through coil 1 is given by

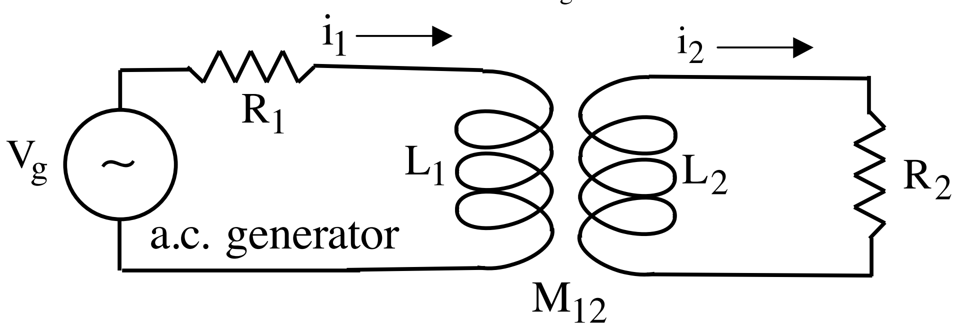
$$\Phi_1 = L_1 i_1 + M_{12} i_2$$

and that for coil 2 by

$$\Phi_2 = L_2 i_2 + M_{21} i_1$$

where the mutual inductances, M_{12} and M_{21}, satisfy the relation $M_{12} = M_{21}$. Collectively these elements make up what is called a transformer.

(a) Suppose you are given the circuit shown below involving two resistors, R_1 and R_2, and a generator producing a constant voltage V_g, together with L_1, L_2, and M_{12}.



(a) If an a.c. current i_1 of angular frequency ω flows through R_1, use Faraday's flux law and Kirchoff's circuit laws to calculate the current i_2 flowing R_2.

(b) Consider the limit $\omega L_1, \omega L_2 >> R_1, R_2$ and assume $M_{12} = \sqrt{L_1 L_2}$. What must the ratio L_1 / L_2 be in order that the power dissipated in the resistor R_1 be the same as that in R_2 (this condition is called impedance matching).

(c) If the turns that make up an inductor are such that they all have the same area, and all of the flux produced by any one turn flows through the remaining turns,[1] how would you expect the total flux, and hence the inductance, to scale with the total number of turns, n in the coil.

(d) From this scaling law obtained in part (c), how would the ratio L_1 / L_2 in the above transformer scale with the turns ratio n_1 / n_2.

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[1] Note this can be achieved by having a very high permeability "core" on which the coil is wound. In practice ferrite materials are used for this purpose.

Problem 4

In an electric field a particle will acquire an induced dipole moment. Assume you are given a spherical particle with radius, a, having specific dielectric constant, κ in a constant electric field, **E**. Calculate its dipole moment.