Northwestern University Physics Qualifying Examination

Wednesday, September 19, 2007

Quantum Mechanics

Solve 3 out of 4 problems

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

1. NMR/Selection Rules

A solid contains nuclei of spin 3/2. In the presence of a static magnetic field H_0 along the z-axis, the Hamiltonian for a spin is

$$\mathcal{H} = \eta\left(S_z^2 - \frac{5}{4}\right) - \hbar\gamma H_0 S_z$$

where η and γ are positive constants, and S_z is in units of $\hbar$.

(a) Draw a single graph showing all the energy levels as a function of H_0. Mark and label your drawing informatively, showing the relevant quantum number(s).

(b) A weak rf magnetic field at frequency ω_0 is now applied along the x-axis. Keeping ω_0 fixed, the field H_0 is slowly swept (varied). Find the values of H_0 at which significant energy absorption takes place.

(c) Explain any selection rule(s) you used in part (b).

2.Landau Levels

Consider an electron moving in two dimensions (the *xy* plane). A magnetic field B is applied along the z-axis.

(a) Ignoring spin, and working in the gauge

$$A_x = 0, \quad A_y = Bx,$$

show that the energy eigenfunctions may be taken in the form

$$\psi(x, y) = e^{iky}\chi(x) \ ,$$

where $\chi(x)$ stands for some function of x alone.

(b) Show that the eigenvalue problem for $\chi(x)$ is that of a harmonic oscillator. Find the angular frequency, ω, and the equilibrium position, x_0, of the oscillation. How does x_0 depend on k?

Note: Do not solve for the eigenfunctions explicitly. There is no need to write Hermite polynomials or even raising and lowering operators, although if it helps you to do so, please do so.

(c) Labeling the different harmonic oscillator states by n, give $E_{k,n}$, the energy of a state with wavevector k, and oscillator excitation state number n.

(d) Find the degeneracy of the ground state, given that the electron is confined to a large rectangular region of dimensions L_x and L_y. Ignore edge effects. Your answer should be proportional to the area $L_x L_y$.

3. Scattering from 1-dimensional delta-function barrier

(a) Consider particles of mass m, incident from the left on the δ-function barrier

$$V(x) = V_0 \delta(x); \qquad V_0 > 0.$$

Writing the incident wave function as e^{ikx}, and the transmitted wave as te^{ikx}, find the transmission amplitude t in terms of k and the parameters of the problem. (Hint: Integrate Schrodinger's equation from $-\varepsilon$ to ε where ε is infinitesimal.)

(b) Find the ratio of the transmitted flux to the incident flux.

(c) Now suppose the particles to have spin ½, and the barrier to be spin-dependent in the following way. If the incident particles have spin up, the reflected and transmitted particles also have spin up, and the transmission amplitude is $t_\uparrow$. Likewise, if the incident particles have spin down, reflected and transmitted particles also have spin down, and the transmission amplitude is $t_\downarrow$.

(d) Consider an incident beam of particles in the spin state

$$\tfrac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right).$$

What is the expectation value $\langle S_z \rangle$ for the transmitted particles, given $|t_\downarrow|^2 = 3|t_\downarrow|^2$?

4. Bound states and scattering from spherical well

(a) Consider a particle of mass m in a spherically symmetric potential

$$V(r) = \begin{cases} -V_0, & 0 \le r \le r_0 \\ 0, & r > r_0, \end{cases}$$

Find V_0 such that there is only one bound state in the well at an energy infinitesimally below zero.

(b) Consider scattering of very low energy particles from the well with the value of V_0 found in (a). Find the s-wave scattering phase shift, δ_0.

Useful Information

$$-\nabla^2 f(r) Y_{\ell m}(\vartheta,\varphi) = \left[-\frac{1}{r}\frac{d^2}{dr^2}\left(rf(r)\right) + \frac{\ell(\ell+1)}{r^2} f(r)\right] Y_{\ell m}(\vartheta,\varphi)$$

$$e^{ikr} = \frac{\sin kr}{4\pi kr} + \text{ non-}s\text{-wave terms.}$$

If the asymptotic wave function for a scattering state is written as

$$\psi(r) \approx \frac{e^{i\delta_0}}{kr}\sin(kr+\delta_0) + (\ell \ge 1 \text{ terms}), \quad (r \to \infty)$$

then δ_0 is the scattering phase shift.