Northwestern University Physics Qualifying Exam
Wednesday, June 6, 2007

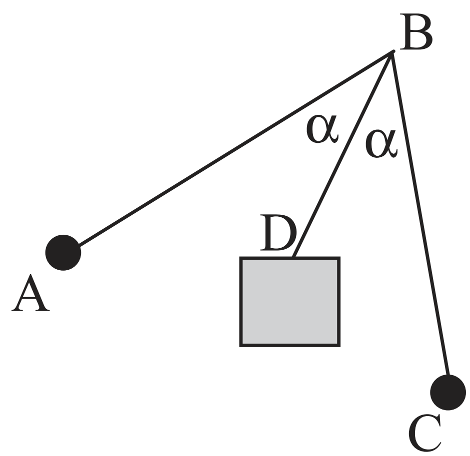
Classical and Statistical Mechanics

This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the exam for a total of four problems.

Do each problem in a separate blue book and write your ID number (not your name) and the problem number on each book. If your solution uses more than one book, label each book with 1 out of 2, 2 out of 2...

Classical Mechanics - do 2 out of 3 problems

1. A rigid structure consists of three massless rods joined at a point B. Two point masses (each of mass m) are attached to the two ends A and C as shown. The lengths $AB = BC = L$ and $BD = l$. The angle $ABD = DBC = \alpha$. The rigid system is supported at the fixed point D and rocks back and forth with a small amplitude of oscillation in the $x - y$ plane.



   (a) Derive the equation of motion from the Lagrangian equation.

   (b) What is the oscillation frequency?

   (c) What is the maximum length l for which stable oscillations are possible?

2. Two bodies of mass m_1 and m_2 are bound by an central attractive force of the form

$$F(r) = -\frac{k}{r^n}\hat{\mathbf{r}}$$

where $\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r_1}} - \overrightarrow{\mathbf{r_2}}$ and n is an integer.

(a) Find the radius ρ for a circular orbit of angular momentum L.

(b) Write down the conditions for stability of that orbit.

(c) Determine the range of values of n for which such stable circular orbits exist.

3. A raindrop is initially at rest. Its initial mass is m_0. It begins to fall under gravity. As it falls, it absorbs water from the surrounding air. Its mass $m(t)$ increases at the rate $dm/dt = bmv$, where b is a coefficient and $v(t)$ is the velocity. Please neglect air resistance.

   (a) Write the equation of motion for the raindrop.

   (b) Prove that the speed of the raindrop will eventually approach a constant value, and find this constant value.

Statistical Mechanics - do 2 out of 3 problems

1. Consider a rubber band in which the tension force F is directly proportional to both the length increase $L - L_0$ (where L_0 is the equilibrium length), and to the temperature T.

$$F = aT(L - L_0); \qquad \text{where } a \text{ is a positive constant}$$

Also, let the heat capacity C_L measured at $L = L_0$ be given by:

$$C_L = bT; \qquad \text{where } b \text{ is a positive constant}$$

   (a) Write down the appropriate thermodynamic identity for dU, for this system.

   (b) The entropy $S(T, L)$ is a function of both T and L. Knowing $S(T_0, L_0)$ for some temperature T_0, find $S(T, L)$ for any T and L.

   (c) If you start at $T = T_i$ and $L = L_0$, and then stretch the thermally insulated rubber band quasi-statically until it reaches the length L_f, what is the final temperature T_f? Is T_f larger or smaller than T_i?

2. At temperature $T = 25$ degrees C, the vapor pressure of H_2O is about 0.0317 atm. (P = 1 atm is defined as the pressure of the Earth's atmosphere, which is about 1.013×10^5 N/m^2.) Thus, on an imaginary Earth having only water in its atmosphere and having a uniform temperature of 25 C, the atmosphere would consist of only water at pressure $P =$ 0.0317 atm (in equilibrium with the oceans). Now consider what happens when one adds the other gases (N_2, O_2, etc.) to the atmosphere of this imaginary Earth, to bring the total pressure up to 1 atm. The goal of this problem is for you to show that this does not strongly affect the equilibrium concentration of atmospheric water vapor. I.e., the equilibrium partial pressure of water vapor in the Earth's atmosphere is indeed roughly 0.03 atm.

   (a) Let P represent the total pressure, and let P_v represent the equilibrium partial pressure of water vapor. Show that

   $$\frac{dP_v}{dP} = \frac{P_v V_l}{N_A k T}$$

   where V_l is the molar volume of the liquid water, N_A is Avogadro's number and k is Boltzmann's constant. (HINT: For any pressure and temperature, assuming equilibrium conditions, how must the molar Gibbs free energy of the liquid water relate to that of the atmospheric water vapor?)

   (b) Solve this differential equation for P_v and use the resulting expression to make a very rough estimate for the fraction by which the equilibrium water vapor pressure changes due to the presence of the other atmospheric gases. ($N_A \simeq 6 \times 10^{23}$ and Boltzmann's constant $k \simeq 1.4 \times 10^{-23}$ J/K).

3. The heat capacity (at constant volume V) of Yttrium Iron Garnet (YIG), a cubic insulating ferromagnet, at low temperatures is given by

$$C_{\rm V} = \alpha\, T^{\nu} + \beta\, T^{3}\,,$$

where $\alpha, \beta > 0$ and $0 < \nu < 3$. The contribution to the heat capacity that is proportional to T^3 is due to quantized lattice vibrations (*phonons*). However, the dominant contribution to the heat capacity at sufficiently low temperatue origniates from *magnons*, which in analogy to phonons are quantized spin vibrations of the local magnetization relative to the direction of the ferromagnetic moment. They are described by the Hamiltonian

$$H = \sum_{\mathbf{k}}^{|\mathbf{k}|\le k_{max}} \hbar\omega_{\mathbf{k}}\, a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}\,,$$

where $\omega_{\mathbf{k}} = D\,k^2$ is the magnon dispersion relation; D is the spin-wave "stiffness" and $\mathbf{k}$ is the wavevector for the magnon. The magnon creation and annhilation operators obey Bose-Einstein commutation relations,

$$\left[\, a_{\mathbf{k}}\,,\, a^{\dagger}_{\mathbf{k}'} \right] = \delta_{\mathbf{k},\mathbf{k}'}\,.$$

There is a maximum wavevector (minimum wavelength) determined by the lattice spacing, a, i.e. $|\mathbf{k}| \le k_{\rm max} = \pi/a$. There is also a maximum energy for a magnon determined by the energy cost to completely reverse the direction of nearest neighbor spins, i.e. the "exchange energy", J. The spin-wave stiffness is related to the exchange energy and lattice spacing by $D = Ja^2/\pi^2\hbar$.

(a) The number of magnons is not conserved. Calculate the mean (i.e. ensemble average) number of magnons of wavevector $\mathbf{k}$ for a ferromagnet in equilibrium at temperature T.

(b) Calculate the magnon density of states, $\mathcal{N}(\omega)$, i.e. the number of modes per unit volume per unit frequency.

(c) Starting from the partition function calculate the magnon contribution to the specific heat for $k_B T \ll J$.