1. Recall that the Lagrangian (L) for a charged particle interacting with an external electromagnetic field is

$$L = \frac{1}{\gamma}[-m_0 c^2 + \frac{e}{m_0 c} p_\mu A_\mu]$$

where e is the charge, m_0 is the rest mass, γ is the relativistic factor $\sqrt{(1 - v^2/c^2)}$, $A_\mu = (\overrightarrow{\mathbf{A}}, i\Phi)$.

$$\frac{e}{m_0 \gamma c} p_\mu A_\mu = \frac{e}{c} v_\mu A_\mu = \frac{e}{c} \overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{A}} - e\Phi$$

   (a) Lagrange's equations are derivable from Hamilton's Principle of Least Action:

$$\text{Action} \equiv \int_a^b L dt$$

   Show that the action for the given L is Lorentz Invariant.

   (b) Show that the Lagrange Equations lead to the usual equations of motion. Hint: Recall the convective derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \overrightarrow{\mathbf{v}} \cdot \overrightarrow{\nabla}$$

2. A point magnetic moment $\overrightarrow{\mu}$ moves through a static magnetic field $\overrightarrow{\mathbf{B}}$ with velocity $v << c$.

   (a) Write the equation of motion for the magnetic moment. It is conventional to write
   $$\overrightarrow{\mu} = g\frac{e}{2mc}\overrightarrow{\mathbf{s}}$$
   where $\overrightarrow{\mathbf{s}}$ is the "spin" angular momentum for the system.

   (b) Show that $\overrightarrow{\mu}$ precesses around the local direction of $\overrightarrow{\mathbf{B}}$ and give an expression for the precession angular frequency ω.

   (c) Consider a solenoid with N turns, current I, radius a and length d. What is the $\overrightarrow{\mathbf{B}}$ field at distances $r >> a, d$ from the center of the solenoid?

   (d) Suppose a particle with magnetic moment $\overrightarrow{\mu}$ passes down the symmetry axis of the solenoid from $-L$ to $+L$, with $L \gg a, d$. Derive a formula for the net rotation of $\overrightarrow{\mu}$ due to precession in the local field of the solenoid. Hint: Use one of Maxwell's equations in integral form directly.

3. Write down Maxwell's equations for homogeneous media with dielectric constant ϵ, $\mu = 1$ and conductivity σ where $\overrightarrow{\mathbf{J}} = \sigma \overrightarrow{\mathbf{E}}$.

   (a) Derive the wave equation for plane wave propagation of $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{B}}$ and show that the wavevector ($\mathbf{k}$) and frequency (ω) are related by

   $$k^2 = \frac{\epsilon\,\omega^2}{c^2}(1 + i\frac{4\pi\sigma}{\omega\epsilon})$$

   (b) Show that in the good conductor limit, $k = \frac{1}{\delta}(1 + i)$ where the skin depth $\delta \equiv \sqrt{2\pi\omega\sigma}$.

4. Recall the multipole expansion of the potential field produced by a localized charge distribution:

$$\Phi(\vec{\mathbf{x}}) = \frac{q}{r} + \frac{\vec{\mathbf{p}} \cdot \vec{\mathbf{x}}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + ...$$

where the quadrupole tensor

$$Q_{ij} \equiv \int (3x'_i x'_j - r'^2 \delta_{ij})\rho(\vec{\mathbf{x}}')d^3x'$$

Consider an axially symmetric ellipsoid with a uniform charge density, total charge q and a boundary surface

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1$$

(a) Q_{ij} has 9 elements. The tensor can be characterized fully by a single parameter, usually taken to be Q_{33}. Explain.

(b) Find Q_{33} in terms of q, a, and b.

(c) Calculate the electric field at distance r away from the center of the ellipsoid along i) the z-axis, and ii) in the xy plane.