1. A particle of mass m is confined by a harmonic potential in one dimension. The restoring force and mass define the natural frequency, ω, of this oscillator, and the non-relativistic Hamiltonian governing its dynamics is:

$$\widehat{H}_0 = \frac{1}{2m}\,\widehat{p}^2 + \frac{1}{2}m\omega^2\,\widehat{x}^2\,. \tag{1}$$

where $\widehat{p}$ is the momentum operator and $\widehat{x}$ is the canonically conjugate position operator.

   (a) The Hamiltonian can be put into dimensionless form,

$$\widehat{H}_0 = \frac{1}{2}\epsilon_0\left(\widehat{\bar{p}}^2 + \widehat{\bar{x}}^2\right)\,, \tag{2}$$

   where $\widehat{\bar{x}} = \widehat{x}/\ell$ and $\widehat{\bar{p}} = \widehat{p}/(\hbar/\ell)$ are dimensionless operators obtained by scaling length in units of ℓ, momentum in units of $\hbar/\ell$, and energy in units of ϵ_0. Determine ϵ_0 and ℓ.

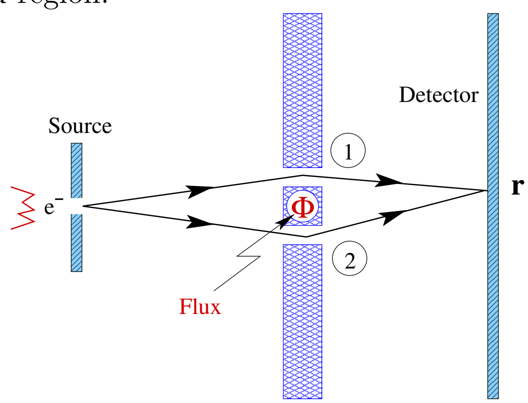
   (b) Show that $a^\dagger = (\widehat{\bar{x}} - i\widehat{\bar{p}})/\sqrt{2}$ and $a = (\widehat{\bar{x}} + i\widehat{\bar{p}})/\sqrt{2}$ are the raising and lowering operators for the quantum oscillator, and that $\widehat{n} = a^\dagger a$ is the operator corresponding to the number of quanta of excitation of the oscillator. Express the Hamiltonian in terms of the raising and lowering operators.

   (c) Calculate the mean momentum and energy of the particle in the ground state of the oscillator.

   (d) Starting from the *relativistic* kinetic energy of a spinless particle of mass m calculate the corrections to the ground state energy of the non-relativistic Hamiltonian to order $(\hbar\omega/mc^2)^2$.

   (e) Calculate the correction to the state vector to first-order in $\hbar\omega/mc^2$ for the ground state.

2. Consider the two-slit electron diffraction geometry shown in the figure. The two-slit geometry includes an *excluded region* of space in which the electron never penetrates. A very long solenoid extends through this excluded region.



   (a) Write down the time-dependent Schrödinger equation for the electron amplitudes for the case of zero flux in the excluded region.

   (b) Let $\psi_{01}(\mathbf{r})$ ($\psi_{02}(\mathbf{r})$) be the probability amplitude for detecting an electron at $\mathbf{r}$ given it passed through slit 1 (2) when there is zero current in the solenoid. What is the resulting probability for detecting an electron at $\mathbf{r}$ if both both slits are open?

   (c) Now consider the same geometry, but with magnetic flux Φ penetrating the excluded region of space. Write down the time-dependent Schrödinger equation for the electron amplitudes for this case.

   (d) Consider a point $\mathbf{r}$ corresponding to a bright fringe in the interference pattern for the case of zero flux in the solenoid. Calculate the change in the intensity at this point when there is flux Φ in the forbidden region.

   (e) For special values of the flux Φ the bright fringe becomes a dark fringe. Calculate these values in terms of fundamental constants, e, $\hbar$, c, m, etc.

3. A spin 1/2 particle with gyromagnetic ratio γ is prepared in a spin state maximally polarized along the $+x$ direction perpendicular to a magnetic field $\mathbf{B} = B\,\hat{\mathbf{e}}_z$. Assume the particle is stationary.

   (a) What is the Hamiltonian describing the motion of the spin in a magnetic field? What are the energy eigenvalues and associated eigenvectors for the spin in the applied field?

   (b) Express the initial state in terms of a linear combination of eigenstates of the Hamiltonian.

   (c) Construct the time evolution operator in the basis of energy eigenstates.

   (d) In the Schrödinger picture the base kets are fixed, and the state vector evolves in time. Find the state vector $|\,\psi(t)\,\rangle$ at time $t > 0$ by applying the time-evolution operator. What is the minimum time required for the spin state to return to the orginal state?

4. Consider the scattering of visible light off free electrons in the non-relativisitic limit.

   (a) Write down the Hamiltonian for the electron in the presence of the radiation field. Use the transverse gauge.

   (b) Write down the vector potential in the transverse gauge in terms of the creation and annihilation operators for photons with wavevector $\mathbf{k}$ and helicity λ.

   (c) For the scattering process what is the relevant interaction Hamiltonian between the electrons and photons?

   (d) Calculate the total cross-section for photons in the visible range of wavelengths scattering off non-relativistic free electrons. Express your result in terms of fundamental constants and the charge and mass of the electron.