Problem 1: Galactic Rotation Curves

(a) Consider a uniform thin spherical shell of total mass M and radius a. Ignoring the thickness of the shell, compute the Newtonian potential at a distance R from the center of the shell. Consider both $R > a$ and $R < a$.

(b) Consider a galaxy made of a large number of stars rotating on circular orbits with respect to the center of the galaxy. Assuming the stars inside the galaxy form a spherical and uniform distribution of radius $\mathcal{R}$, which is a good approximation observationally, compute the circular orbital velocity v of the star as a function of r, the distance to the galactic center. Draw a schematic diagram for $v(r)$, which is called the galactic rotation curve, for both $r > \mathcal{R}$ and $r < \mathcal{R}$.

(c) Figure 1 is an example of the measured galactic rotation curve, where the curve flattens as the distance to the center of the galaxy becomes larger than $\mathcal{R}$. (There are isolated stars outside of the galaxy, whose rotation curve could still be measured.) Assuming Newtonian gravity, what is the mass density of the galaxy as a function of radius? How can you reconcile this mass distribution with the observed spherical and uniform distribution of the visible stars?

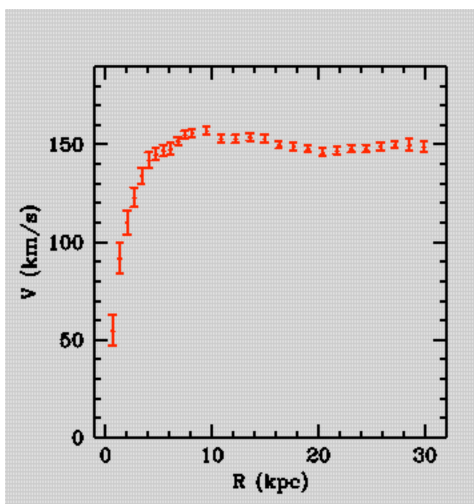

FIG. 1: *The rotation curve for the galaxy NGC3198 from Begeman 1989.*

Problem 2: Path of Quickest Descent

(a) Consider a bead sliding without friction on a wire of some shape in the vertical (x, y) plane in the presence of constant gravitational acceleration g in the y-direction. Assuming the bead is released at rest from the origin $(0, 0)$, what is the velocity of the bead at position (x_0, y_0)?

(b) Suppose the wire has the shape $y = y(x)$, use your result in (a) to write down the integral for the total time it takes for the bead to travel from the origin to (x_0, y_0).

(c) If $y(x)$ is such that the total time for the trip between $(0, 0)$ and (x_0, y_0) is minimum, derive the differential equation satisfied by $y(x)$.

(d) Now consider there is a sliding friction on the wire $f_s = \mu F_N$, where F_N is the magnitude of the normal force exerted on the wire. derive the differential equation satisfied by $y(x)$.

Problem 3: Foucault's Pendulum

(a) Consider a non-inertial moving frame rotating with angular velocity $\vec{\omega}$ with respect to a fixed frame. Given an arbitrary vector $\vec{A}$, show that the time derivative $d\vec{A}/dt$ in the fixed frame is related to $\delta\vec{A}/\delta t$, the time derivative of $\vec{A}$ relative to the rotating frame, as follows:

$$\frac{d\vec{A}}{dt} = \frac{\delta\vec{A}}{\delta t} + \vec{\omega} \times \vec{A}.$$

(b) For a particle with mass m moving near the surface of the earth, we can choose a coordinate system S that is fixed on the earth's surface, which rotates with a constant angular velocity $\vec{\omega}_e = 2\pi/\text{day}$ relative to an inertial frame S_I fixed in the space. Show that the equation of motion for the particle with position $\vec{r}$ inside the frame S is

$$m\frac{\delta^2\vec{r}}{\delta t^2} = \vec{F} - m\vec{g} - m\left[\vec{\omega_e} \times (\vec{\omega_e} \times \vec{r}) - 2\omega_e \times \frac{\delta\vec{r}}{\delta t}\right],$$

where $\vec{g}$ is the gravitational acceleration near the earth's surface and $\vec{F}$ is any non-gravitational force acting on the particle.

(c) The Foucault's Pendulum is a simple pendulum which can oscillate a long time without being appreciably damped by friction. Compute the precession frequency of the Foucault's Pendulum at a latitude θ above the equator, near the earth's surface, assuming the pendulum's frequency is much larger than the earth's rotation frequency.