POTENTIALLY USEFUL EQUATIONS

Maxwell's Equations, in MKS units:[1]

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0},$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$
$$\vec{\nabla} \cdot \vec{B} = 0,$$
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t},$$

where $\epsilon_0 = 8.854 \times 10^{-12}$ $\mathrm{C^2/(N{\cdot}m^2)}$, $\mu_0 = 4\pi \times 10^{-7}$ $\mathrm{N/A^2}$. Note that $\mu_0\epsilon_0 = c^{-2}$, where $c = 299,792,458$ m/s is the speed of light in vacuum. Here, ρ and $\vec{J}$ are, respectively, the charge and current density.

Conservation of Charge:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0.$$

Lorentz Force Law (force on point particle with charge q and velocity $\vec{v}$), in MKS units:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right).$$

[1]Throughout, whenever there is a choice to be made, MKS units have been used for the electric and magnetic fields.

(1) **Neutral Pion Decay** – Neutral pions (π^0) are spin-zero particles copiously produced in high energy collisions. Neutral pions decay, 98.8% of the time, into two photons, $\pi^0 \rightarrow \gamma\gamma$. The neutral pion rest mass is m_π and the photon is massless. While addressing the questions below, feel free to use natural units, where the speed of light in vacuum is set to unity ($c = 1$).

(a) In the laboratory, a neutral pion is observed to be moving along the x-axis with energy E_π. Compute its velocity $\vec{\beta}$. If the pion lifetime (at rest!) is τ_π, how long does it live in the lab?

(b) In the lab, after the pion decays, what are the smallest and largest possible decay photon energies, $E_\gamma^{\min}$ and $E_\gamma^{\max}$ respectively?

(c) In the pion rest frame, the photon angular distribution is isotropic, *i.e.*,

$$\frac{1}{N_\gamma}\frac{dN_\gamma}{d\Omega} = \frac{1}{4\pi},$$

where Ω is the solid angle in the pion rest frame and N_γ is the number of photons. Use this information to compute the photon energy distribution in the reference frame defined in (a). Sketch your answer.

(d) In the lab, what are the largest and smallest possible opening angles ($\theta_{\max}$ and $\theta_{\min}$, respectively) defined by the two decay photons?

(2) **Ion Trapping** – The ability to trap ions is of fundamental importance in experimental physics. As these are charged one can often do this by judiciously building electromagnetic field configurations. For the purposes of the questions below, an object is considered trapped at a position $\vec{r}$ when it is, dynamically speaking, in *stable equilibrium* at $\vec{r}$.

(a) (One form of) Earnshaw's theorem states: "A charged particle cannot be held in stable equilibrium by electrostatic forces alone," which means that it is impossible to trap a charged particle at rest in free space using a static distribution of charges. Prove Earnshaw's theorem.

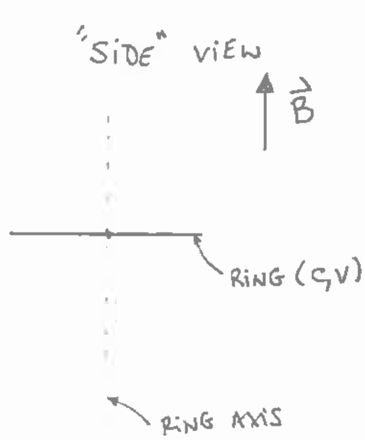
A simple electromagnetic trap can be built by placing a conducting ring (with capacitance C) of radius a in a constant magnetic field $\vec{B}$ and charging the ring it by raising it to a voltage V. The axis of the ring is oriented parallel to the magnetic field, as depicted in the figure.

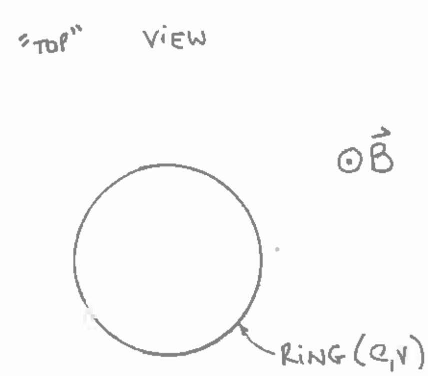
(b) The equilibrium position of this trap is at the center of the ring. Show that the equilibrium is stable when it comes to perturbing an ion's position along the ring's axis as long as the ion's charge is opposite to the charge stored in the ring. What is the frequency of small oscillations along this direction as a function of a, V, C, q (the charge of the ion) and m (the mass of the ion)?

(c) If the ion is confined to move along the axis of the ring, what is the maximal velocity it can have at the center of the ring and still remain trapped?

(d) For positions very close to the equilibrium point, compute the component of the electric force acting on the ion *parallel* to the plane of the ring assuming that the ion's charge is opposite to that of the trap.

(e) As far as motion in the plane of the ring is concerned, the same ion can get "trapped" because of the magnetic field, if the magnetic field is strong enough. Write down the equation of motion assuming the ion is constrained to move in the plane of the ring. Show that one solution to the equation of motion of the ion is a circular orbit around the equilibrium position. Compute the orbital frequency. How strong a magnetic field you need (compared to a combination of a, V, q, C, etc) in order to "trap" the ion in this fashion?





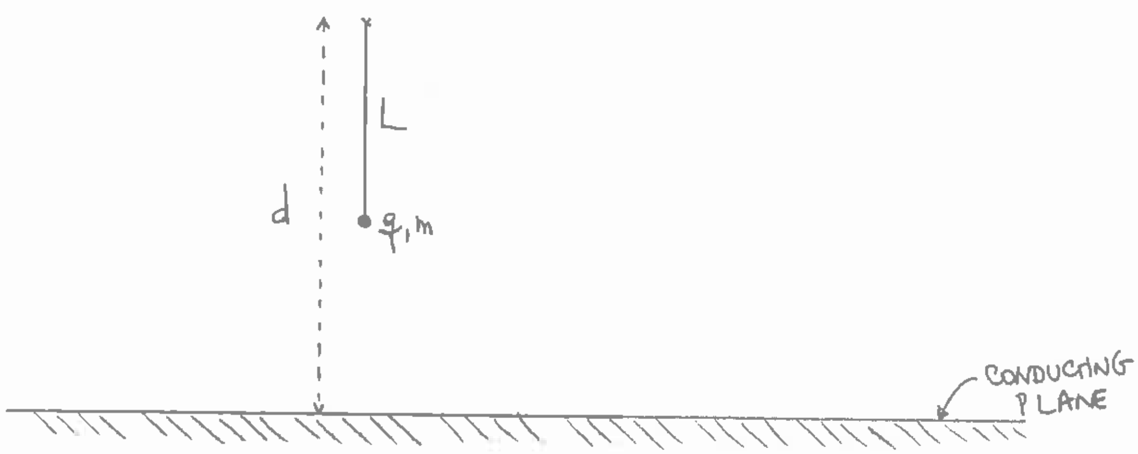
(3) **Charged Pendulum and Infinite Conducting Plane** – A particle of mass m and charge q is suspended on a string of length L. A distance $d > L$ under the point of suspension there is an infinite perfectly conducting plane.

(a) Compute the force (including its direction) between the point mass and the plane when the point mass is at rest, closest to the infinite conducting plane.

(b) Compute the frequency of the pendulum assuming the amplitude of oscillations is small. Assume that the force between the conductor and the point charge is much larger than the force of gravity.

(c) Since the particle is accelerated, it will radiate energy off to infinity in the form of electromagnetic radiation. Intuitively, one should suspect that the power P radiated will be proportional to the oscillation frequency ω to some power p, *i.e.*, $P \propto \omega^p$. Qualitatively, what do you anticipate p to be? Why?

(d) Compute how much energy the point mass loses by electromagnetic radiation per unit time if it oscillates with small amplitude a.



(4) **Electromagnetic Wave Hitting a Conductor** - Inside a neutral Ohmic conductor, the free current density is proportional to the electric field:

$$\vec{J} = \sigma \vec{E},$$

where $\vec{E}$ is the electric field and σ is the conductivity of the material.

(a) Show that, inside the conductor, the electric and magnetic fields allow for plane-wave solutions of the type

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \ \vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}.$$

What is the relation between the wave-number $|\vec{k}|$ and the frequency ω? Feel free to concentrate on plane-waves "moving" in the x-direction so that $\vec{k}\cdot\vec{x} = kx$. Assume that the conducting medium is a linear medium characterized by a permeability μ and a permittivity ϵ.

(b) In (a), you should have found that k has an imaginary part (for real ω and σ). What does this mean? Show that, in the limit $\sigma \ll \omega\epsilon$ (poor conductor), the imaginary part of k does not depend on ω.

(c) Imagine that half of the space is filled with a conducting material with constant conductivity σ, as depicted in the figure. Assume that, other than the conductivity σ, the conducting medium has the same properties as the vacuum, *i.e.*, $\mu = \mu_0$, $\epsilon = \epsilon_0$. A polarized plane-wave (electric field in the positive y direction, frequency ω), propagating along the positive x direction hits the conducting surface. Compute the reflection coefficient (ratio of reflected intensity to the incident one), R. What happens in the limits $\sigma \to \infty$ (perfect conductor) and $\sigma \to 0$ (very poor conductor)?

(d) Assuming the medium in item (c) is a poor conductor ($\sigma \ll \omega\epsilon$), what is the transmitted intensity T as a function of the wave's distance from the origin? Is $R + T = 1$? Explain.