QM Problem 1

A Hamiltonian is supersymmetric if it can be written as $\mathbf{H} = \mathbf{Q}^2$ where Q is Hermitian.

(a) Show that the eigenvalues of H are non-negative.

Next consider a supersymmetric harmonic oscillator with

$$\mathbf{H} = (\mathbf{p}^2/2m)\,\mathbf{1} + \tfrac{1}{2}m\omega^2x^2\,\mathbf{1} - \tfrac{1}{2}i\hbar\omega\,\sigma_1\sigma_2$$

where $\mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Note: $[\sigma_i, \sigma_j] = 2i\sigma_k$ $\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1}$ $\mathbf{p} \equiv -i\hbar\partial/\partial x$

(b) Find an explicit $\mathbf{Q}$ constructed from the σ_i, p and x such that $\mathbf{H} = \mathbf{Q}^2$

(c) What is the lowest eigenvalue and coresponding eigenfunction? (There's no need to normalize the eigenfunction. Note that $\psi(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$

(d) What is the degeneracy of the eigenvalues of **H**?

QM Problem 2

A particle is trapped in a short range potential well having a single bound state with energy $-E_o$. (You can consider this to be an atom that has no relevant excited states.)

A second well identical to the first is at a large distance *b* from the first, so that there are two eigenstates $-E_o \pm F$ where $|F| \ll |E_o|$. A third well is added at a distance *b* from the other two, forming an equilateral triangle. We will use a basis in which $\begin{pmatrix}1\\0\\0\end{pmatrix}$ means that the particle is purely in the first well, etc.

a) If there are only two wells, write the 2x2 matrix for the Hamiltonian.

b) For the three-well system, find the energies of the lowest three eigenstates *to first order* in F.

c) If the system is initially in the first well, i.e. $\psi(t=0) = \begin{pmatrix}1\\0\\0\end{pmatrix}$, find the probability that it will be in the second well at time t, to lowest order in t (i.e. t is small compared to $\hbar/F$....write $t=\varepsilon\hbar/F$ where $\varepsilon \ll 1$)

QM Problem 3

Consider an atom that has one ground state |g> and one excited state |e>. The Hamiltonian for the system is

$$H_{atom} = \tfrac{1}{2}\hbar\omega_o \left(|e\rangle\langle e| - |g\rangle\langle g|\right)$$

The atom is in an electric field that is represented in this problem as a collection of photons with

$$H_{photon} = \hbar\omega_o \left(a^\dagger a + \tfrac{1}{2}\right) \text{ where } [a^\dagger, a]=1.$$

[Note that there is only a single mode.]

The interaction between the two subsystems corresponds to absorption/emission of a photon coupled with excitation/deexcitation of the atom, governed by a coupling constant χ:

$$H_{int} = \chi \left(a|e\rangle\langle g| + a^\dagger|g\rangle\langle e|\right)$$

(a) If the atom is in its ground state |g> at time t=0, with *n* photons in the field, what is the probability that is in the ground state at some nonzero t?

[You may want to write the combined state of the ground-state atom and n photons as |g, n> and the combined state after photon absorption as |e, n-1>.]

(b) Next consider a coherent field of photons, described by

$$\left|\alpha\right\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \left|n\right\rangle$$

Is such a coherent state an eigenfunction of a and/or $a^\dagger$? If yes, what are the eigenvalues?

(c) If the atom in its ground state is placed in this coherent field at *t*=0, what is the probability that the atom is still in the ground state at some nonzero *t*?

QM Problem 4

Consider a charged particle in a magnetic field. This problem must be worked out in a gauge such that $\vec{A} = -By\hat{x}$ where B is a constant. Recall that the Hamiltonian is $H = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2$.

(a) Find $[\vec{p}, H]$.

(b) Use (a) to show that the eigenstates of H are of the form $\psi(\vec{r}) = X(x)Y(y)Z(z)$. What are X(x) and Z(z)?

(c) Use the results of (b) to find a Schrodinger-like equation for Y(y) and thus find the eigenvalues of H. Hint: a coordinate shift in the y direction might be helpful in making the equation for Y(y) simple.