Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Friday, June 12, 2009, 9 am - 1 pm

Classical and Statistical Mechanics

This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the examination for a total of four solved problems.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Classical Mechanics - do 2 out of 3 problems

Problem 1: Gravity Assist

Here we would like to find out how encounters of a spacecraft named Cassini with a nearby planet could be used to gain speed for the spacecraft. We define two (approximate) inertial frames: the planetocentric system has its origin at the center of the planet and the heliocentric system, at the center of the Sun.

(a) First consider the elastic head-on collision of Cassini with an initial speed V in the $+\hat{x}$ with a planet moving at a speed V_p in the $-\hat{x}$. Realistically, the mass m of Cassini is much much smaller than the planet mass M. Find Cassini's speed after the collision.

(b) No long assuming head-on collision, draw the vector diagram relating the initial and final velocity vectors of Cassini in the two frames and argue Cassini gains speed if it passes the planet from behind and loses speed if it passes in front. For now neglect the size of the planet. What is the maximum possible boost in the speed? Under what condition can it happen?

(c) Next consider Cassini swings around a planet following a Keplerian orbit

$$r(\theta) = \frac{h^2}{GM}\frac{1}{1+\epsilon\cos\theta}, \qquad \epsilon = \sqrt{1+\frac{2\mathcal{E}^2h^2}{G^2M^2}},$$

where h and $\mathcal{E}$ are Cassini's angular momentum per unit mass and energy per unit mass, respectively. Furthermore, the planet has a finite radius R. What is the minimum impact parameter required to avoid Cassini crashing into the planet? Is it possible to achieve the maximum possible boost now that the planet has a finite size?

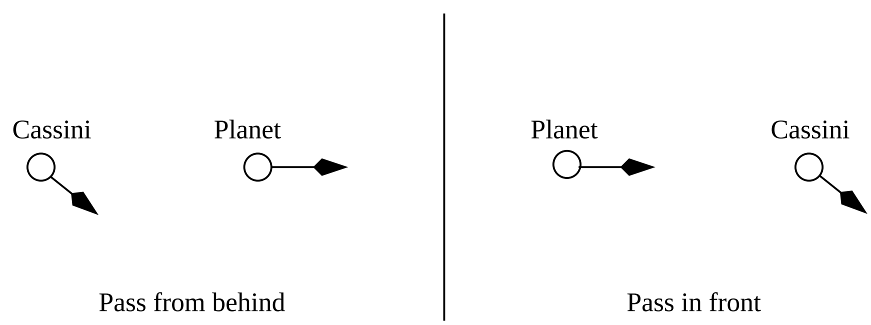


Figure 1

Problem 2: Physics of Billiard Collision

In this problem we will consider the elastic collision of a moving "cue" ball with an identical, stationary "object" ball. (The cue ball is struck horizontally by a "cue" stick. See Figure 2 (a).)We assume the billiard ball is a perfect solid sphere with a radius R and a mass M.

(a) First consider shots in which the cue hits the cue ball directly above the center of the ball. See Figure 2 (a). At what height h should you hit to get a rolling shot (such that the cue ball rolls without sliding)?

(b) Now assume the impact parameter of the collision is b and the cue ball is rolling without sliding. For now assume there is no sliding friction at all. Compute the final velocities $\vec{v}_o$ and $\vec{v}_c$ of the object ball and the cue ball, respectively. Verify the "90-degree rule": $\theta_o + \theta_c = \pi/2$. See Figure 2 (b).

(c) Finally consider there is a sliding frictional force F_f between the surface and the ball, but not between the balls. Do you still have the "90-degree rule"?

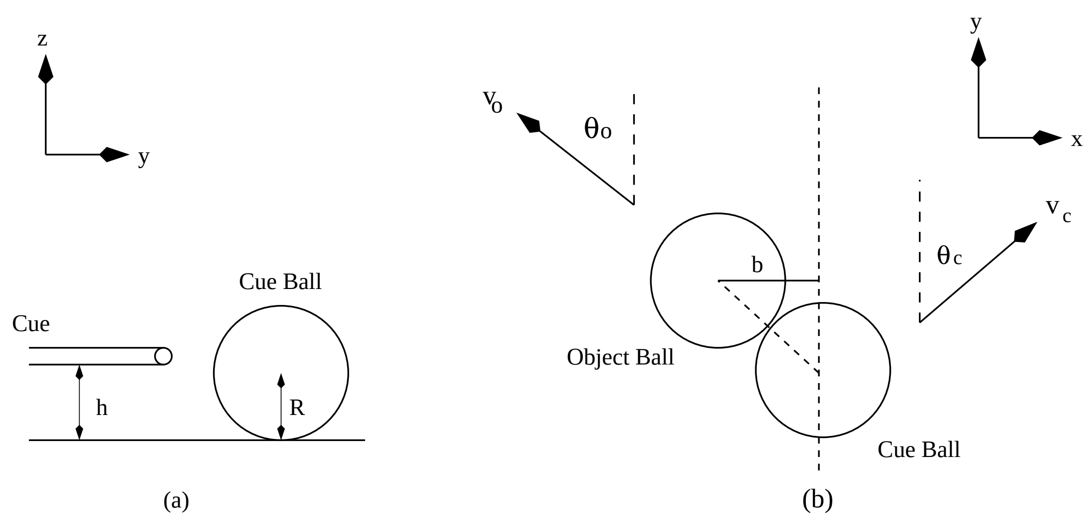


Figure 2

Problem 3: Vibrations of Linear Triatomic Molecules

Consider a linear tri-atomic molecule in which the central atom has mass M and the two symmetrical boundary atoms have masses m. The equilibrium distance between the central and each boundary atom is b.

(a) Without solving for the equation of motion, how many vibrational degrees of freedom are there in this system? Describe what motions correspond to these vibrational modes.

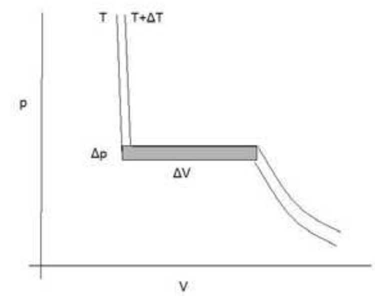
(b) Now consider only vibrations occurring in the plane. Derive the equations of motion of the system using the Lagrangian method.

(c) Determine the eigenfrequencies and describe the normal mode motions.

Statistical Mechanics - do 2 out of 3 problems

Problem 1

The graph at right shows two isotherms for an unspecified material at slightly different temperatures, in the neighborhood of a liquid-vapor phase transition.



Consider a Carnot cycle with 1 mole of this material between the temperatures T and T+ΔT, around the region shown shaded in the diagram. (Since ΔT is small, the exact shape of the isentropic "sides" does not matter, and the cycle can be taken to be a rectangle in the p-V plane.)

(a) Using the Carnot cycle result $\frac{W}{Q} = 1 - \frac{T_1}{T_2}$, derive the Clausius-Clapeyron relation $\frac{dp}{dT} = \frac{L}{T\,\Delta V}$, where T is the phase transition temperature (boiling point) at a pressure p, and L is the latent heat of evaporation. (Note that when ΔT is infinitesimal, so is Δp, but ΔV is not.)

(b) By pumping on liquid helium (reducing the pressure), one can cool it to lower temperatures. At a standard pressure p_o, the boiling temperature is known to be T_o (e.g. 4.2K at 1 atm). Determine the pressure dependence of the boiling temperature, i.e. find an expression for T in terms of p, p_o, T_o, L and universal constants, assuming that (i) L is independent of temperature, (ii) the volume of a mole of liquid helium is negligible compared to the volume of a mole of helium gas, and (iii) helium vapor is an ideal gas. Verify from your result that pumping does lead to cooling.

Problem 2

Hydrogen can be assumed to be an ideal gas. Molecular hydrogen comes in two forms:

(i) Ortho: total spin =1, so spin wave function is symmetric, so rotational wave function must be *antisymmetric,* i.e. only even values of the angular momentum quantum number l are allowed

(ii) Para: total spin=0, so rotational wave function must be *symmetric,* etc. etc.

The molecule has moment of inertia I, so the rotational energy is $\hbar^2 l(l+1)/2I$. The translational partition function is the same for both types, so it factorizes out and need not be considered in this problem.

(a) Write down the partition function for hydrogen as a sum for ortho- terms and another sum for para-terms. Hence write down the probability P_{ortho} that hydrogen molecule is ortho, and similarly write P_{para}. Use the shorthand $T_o \equiv \hbar^2/2k_BI$.

(b) At very high temperature T ($>>T_o$), obtain a simple result for the equilibrium ratio $N_{ortho}:N_{para}$ in hydrogen gas by making the appropriate approximations.

(c) Determine the same ratio when the gas is cooled down to a very low temperature ($T<< T_o$), again making appropriate approximations.

Problem 3

Consider a free electron gas (ideal gas of spin-half fermions with charge –*e*).

(a) Determine the density of states D(ε), defined by $n = \int \frac{D(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon$

Next suppose that a weak magnetic field B is applied in the z-direction, so that electron energies are shifted by $\pm\frac{1}{2}g\mu_B B \approx \pm\mu_B B$.

(b) Write integral expressions for the number densities of spin-up and spin-down electrons in the presence of the magnetic field, assuming that *the density of states is unaltered*. By expanding to first order in B, determine how the two number densities change from the B=0 value. Your answer should be in terms of $\partial n/\partial\mu$.

(c) Calculate the magnetic moment per unit volume M, and thus the magnetic susceptibility per unit volume $\chi \equiv \partial M/\partial B$.

(d) Given the Sommerfeld expansion

$$n = \int_0^{\mu} D(\varepsilon) d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 D'(\mu) \ldots$$

determine only the *first* term in $\partial n/\partial\mu$, and thus the leading term in χ at low temperatures.