Department of Physics & Astronomy
Qualifying Exam
Quantum Mechanics

Problem 1

(a) You are given the one dimensional delta function potential

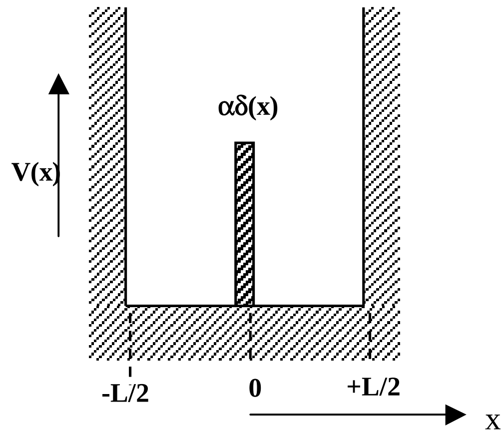
$$V(x) = \alpha\delta(x)$$

where α is a constant. Show that the effect of this potential on the wave function $\psi(z)$ of a particle is such that $\psi(x)$ is continuous at x = 0, but that the slope changes according to the relation

$$\psi'(+\varepsilon) = \psi'(-\varepsilon) - \frac{2m\alpha}{\hbar^2}\psi(x)$$

where ε is an infinitesimal.

(b) You are given the potential shown below



The potential well is infinite, and the rectangle at x = 0 represents the δ - function potential in part (a). Show that the energy levels follow from the solution of the equation

$$k\frac{\cos\left(\pm k\frac{L}{2}\right)}{\sin\left(\pm k\frac{L}{2}\right)} = -\frac{m\alpha}{\hbar^2}$$

where $E = \dfrac{\hbar^2 k^2}{2m}$.

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Problem 2

Two particles of spin ½ and magnetic moments μ_1, μ_2 ($\mu_1 > \mu_2 > 0$) are in an external magnetic field of strength B.

(a) Suppose the Hamiltonian is

$$H_0 = -2(\mu_1\vec{S}_1 + \mu_2\vec{S}_2) \bullet \vec{B}$$

$\vec{S}_1$ and $\vec{S}_2$ are the spin operators for the particles.

Find the eigen functions and eigen values for this system exactly, and sketch an energy level diagram showing all energy levels as <u>functions of B</u>. Make sure you label the levels with appropriate quantum numbers, and explain these quantum numbers.

(b) Repeat part (a) for

$$H_1 = -2(\mu_1\vec{S}_1 + \mu_2\vec{S}_2) \bullet \vec{B} + 4\lambda\vec{S}_1 \bullet \vec{S}_2 \ ,$$

where $\lambda > 0$.

(c) Professor Much Too Smart proposes a new Hamiltonian

$$H_2 = -2(\mu_1\vec{S}_1 + \mu_2\vec{S}_2) \bullet \vec{B} + 4\lambda' \vec{S}_1 \bullet \vec{S}_2 + 16\lambda''(\vec{S}_1 \bullet \vec{S}_2)^2$$

Show that the level splittings given by H_2 are completely equivalent to those given by H_1, with a modified value of λ. Find this value.

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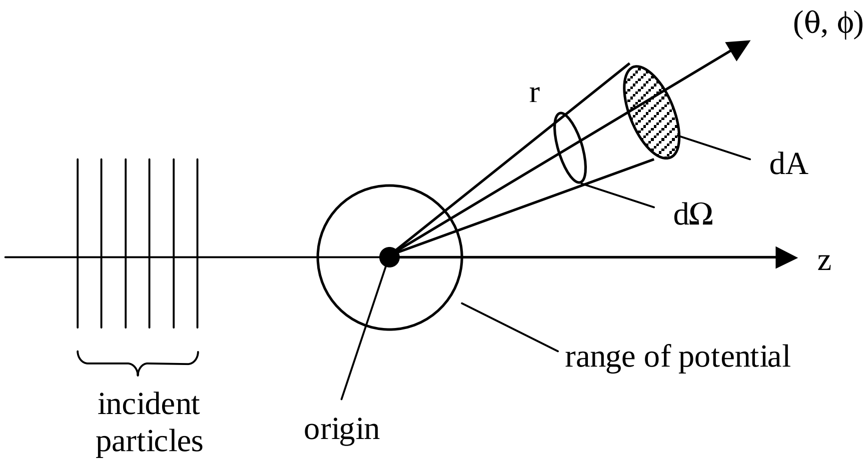
Problem 3

Consider the scattering experiment shown in the figure



Particles traveling along the z axis are scattered by a short-range potential.

(a) If the wave function of the particles at large distances from the potential can be expressed in the form

$$u = \exp(ikz) + \frac{1}{r} f(\theta,\phi)\exp(ikr)$$

show that the differential scattering cross-section is

$$\frac{d\sigma}{d\Omega} = \left| f(\theta,\phi) \right|^2$$

(b) The particles are incident on nuclei of Au, represented by a spherically symmetric potential

$$U(r) = \frac{\beta}{r}\exp(-\gamma r)$$

where β and γ are constants.

Show that, in the Born approximation, the differential scattering cross-section for the scattering vector $\vec{Q} = \vec{k} - \vec{k}'$ is given by

$$\frac{d\sigma}{d\Omega} = \left\{ \frac{2m\beta}{\hbar\left(Q^2 + \gamma^2\right)} \right\}^2$$

It is sufficient to obtain the Q dependence of this result; you need not derive the other factors.

(c) Use this result to derive the Rutherford formula. In other words, show that, for the scattering of α-particles of energy E incident on nuclei of atomic number Z, the differential scattering cross-section for scattering at an angle θ to the incident direction is

$$\frac{d\sigma}{d\Omega} = \left[\frac{Ze^2}{8\pi\varepsilon_0 E \sin^2 \frac{\theta}{2}} \right]^2$$

Department of Physics & Astronomy
Qualifying Exam
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Problem 4

A sample of potassium (nuclear charge Z = 19) has an electron missing from the lowest electronic orbital (the 1S state) of one of the atoms. Shortly thereafter an electron from a higher state makes a transition to fill the empty state.

(a) Calculate the electric dipole moment matrix element, and thus estimate the transition rate from the 2P state to the empty 1S state relative to the frequency of the radiation. You may ignore numerical prefactors, but all non-numerical factors must be obtained. (Dimensional analysis may help.) Would this relative rate be higher or lower if the sample were cesium (Z = 55) instead of potassium? You may approximate the *n*P wave function by $\psi \cong R_{nP}^{-3/2} \cos\theta \; \exp\left(-r/R_{nP}\right)$, where r is the radial co-ordinate, θ is the polar angle, and $R_{nP} = n^2 R_0$. This R_0 is the mean radius of the ground state.

(b) Calculate the transition rate from the *n*P state relative to that from the 2P state, for arbitrary *n*.

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial \rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial \phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial \phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial \rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_2) - \frac{\partial A_1}{\partial \phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial\psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial \phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial \theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial \phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial \theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial\psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial \phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$

SOURCE: JACKSON, CLASSICAL ELECTRODYNAMICS.