**Northwestern University Physics and Astronomy Ph.D. Qualifying Examination**

Friday, June 10, 2011, 9 am - 1 pm

**Classical and Statistical Mechanics**

**This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the examination for a total of four solved problems.**

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with “1 out of 2”, “2 out of 2” and so on.

**Classical Mechanics - solve 2 out of 3 problems**

If you solve all 3 problems, only the first 2 will be graded.

CM1. Garbage disposal in orbit

The space shuttle, on its last flight, is in a circular orbit (radius $R$, angular velocity $\Omega$). At $t=0$ an astronaut tosses a small object (mass $m$) in the direction of the center of the earth with small relative velocity $v_o$. This has negligible effect on the shuttle's orbit.

(a) Describe qualitatively the subsequent motion of the object.

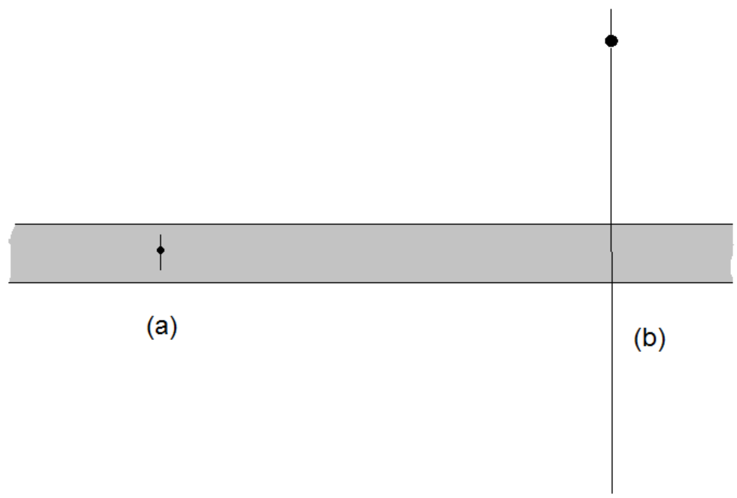
(b) Since $v_o$ is small, write the radius of the object's orbit as $r(t) = R + \delta r(t)$. Find $\delta r(t)$ and the angular velocity $\omega(t)$.

(c) Does the object cross the shuttle's orbit again? Will it ever hit the shuttle?

(d) How would the motion be different if the object were thrown in the opposite direction (away from the earth)?

CM2. Wandering through the galaxy

A galaxy can be approximated by an infinite slab of continuous material having thickness $T$ and density $\rho$. A particle of mass $m$ has initial velocity $v_o$ perpendicular to the plane of the slab, and moves through the slab without friction or collisions (subject only to gravitational forces).




a) Determine the period of oscillations if the particle is always within the slab.

b) Determine the period of oscillations if the amplitude is much larger than the thickness of the slab.

c) Estimate the period for a body (such as the sun) undergoing small oscillations normal to the galactic plane (case (a) above), using a ballpark density estimate of 1 solar mass (~$10^{30}$kg) per cube with sides of ~6 light years.

CM 3: Suppose the laws of mechanics for a particle are invariant under the transformation

$$\begin{aligned} t' &= \gamma t + \frac{\gamma\beta x}{c} \\ x' &= \gamma x + \gamma\beta\, ct, \end{aligned}$$

where $\gamma = 1/\sqrt{1-\beta^2}$. [1]

(a) Show that this implies the laws of mechanics for the particle are also invariant under the velocity transformation

$$v' = \frac{v + \beta c}{1 + \beta v/c}.$$

(b) The laws of mechanics for the particle are also invariant under translations and rotations, and are time-independent, so the Lagrangian must be a function only of the square of velocity, $L = L(v^2)$. By expanding in small $\beta$, show that the following condition must be satisfied in order for the action $S = \int dt\, L(v^2)$ to be invariant:

$$L + 2\frac{dL}{dv^2}c^2\left(1 - \frac{v^2}{c^2}\right) = constant.$$

(c) Explain why this constant can be set to zero, and derive the Lagrangian up to an overall undetermined multiplicative factor.

(d) Expand the Lagrangian obtained to order $v^2/c^2$, and compare to the non-relativistic Lagrangian for a free point particle. By equating the $v^2/c^2$ terms derive the overall multiplicative factor.

[1] Hint: you don't need to know special relativity to solve this problem.

**Statistical Mechanics - solve 2 out of 3 problems**

If you solve all 3 problems, only the first 2 will be graded.

**1.** Consider a planet of mass $M$ and radius $R$, with an atmosphere made of one type of (spherical) molecule of mass $m$ and radius $a$. The pressure of the atmosphere is $p_0$ at the planet's surface. You may assume the atmosphere to be a classical ideal gas.

Recall that the gravitational potential is of the form $U(r) \propto -1/r$, where $r$ is the radial coordinate for spherical coordinates centered at the center of the planet.

(a) Find the density $\rho(r)$ and pressure $p(r)$, as a function of radial coordinate $r$, assuming that temperature $T$ is $r$-independent.

(b) What is the "thickness" of the atmosphere (the height above the planet's surface at which the density falls off to $1/e$ of its value at the surface)?

(c) For the Earth ($R = 6400$ km, $T = 300K$, $p_0 = 100$ kPa, $g \approx 10$ m/s$^2$), estimate the thickness of (b) in meters, assuming a pure nitrogen atmosphere ($m = 28$ hydrogen atom masses for diatomic nitrogen molecules).

(d) Consider a molecule which is at some height $h = r - R$, travelling upwards with a velocity in excess of the escape velocity from the Earth's atmosphere. Estimate the height $h_{\text{escape}}$ above which the molecule has an appreciable (say 50%) probability of not colliding with any other molecules as it moves upwards.

Hint: Your computations in (d) may be simplified by approximating the gravitational acceleration to be constant. If you use this approximation, include a brief justification for it.

(e) Above the height $h_{\text{escape}}$ computed in (d), molecules which exceed the planet's escape velocity can travel out of the atmosphere and into space, since many of them will not suffer collisions with other molecules on the way. Estimate this "evaporation" height in meters for nitrogen molecules in the Earth's atmosphere. You may take $a \approx 2$ Å(recall 1 Å$=$ $10^{-10}$ m).

Hint: $\ln x \approx 2.3 \log_{10} x$

**2.** Consider an ideal gas of $N \gg 1$ atoms of mass $m$, confined to a volume $V$. The atoms are magnetic, and are spin-1/2 with magnetic moment $\mu$. There is no magnetic field applied to the particles in the volume $V$. You may assume there are no magnetic interactions between the atoms.

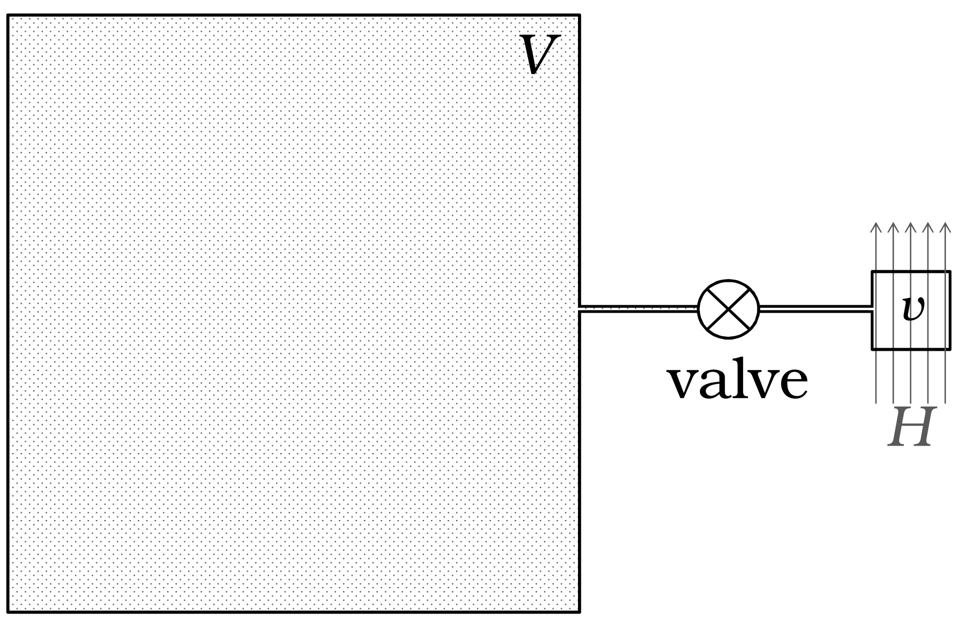
(a) Find the absolute Helmholtz free energy per particle.

Now, a valve is opened allowing the gas to occupy a small chamber of volume $v \ll V$, as well as the large volume. The small chamber is immersed in a constant magnetic field $H$ and is in thermal and particle-exchange equilibrium with the large volume. You may assume that the amount of gas that flows into $v$ does not affect the gas in $V$. You may ignore the volume of the thin pipe connecting $V$ and $v$.

(b) Find the equilibrium number of atoms $n$ found in the small volume as a function of $T$ and external magnetic field $H$.

(c) Find the magnetization of the atoms in the small volume as a function of $T$ and $H$.

(d) Find the density of gas in the small volume for large values of magnetic field (continue to assume that $n \ll N$).



**3.** Consider $N$ spin-1/2, noninteracting fermions which are confined in a system of two two-dimensional boxes, connected by a narrow channel through which particles may pass. The boxes (A and B) each have area $S$. The particles in box B have a potential energy which is $U$ above that of box A. The particles are nonrelativistic and have mass $m$.

(a) Find the Fermi energy $\epsilon_F$ of this system as a function of $U$. Sketch your result.

(b) Find the ground state energy of this system as a function of $U$. Sketch your result.

(c) Find the (two-dimensional) pressure in the gas as a function of $U$ in the low-temperature limit ($k_B T \ll \epsilon_F$). Sketch your result.

**Constants**

$k_B = 1.381 \times 10^{-23}$ J/K $\qquad h = 6.626 \times 10^{-34}$ J·sec $\qquad c = 2.998 \times 10^8$ m/sec
$m_p = 1.673 \times 10^{-27}$ kg $\qquad m_n = 1.675 \times 10^{-27}$ kg $\qquad m_e = 9.109 \times 10^{-31}$ kg
$e = 1.602 \times 10^{-19}$ C $\qquad G = 6.67 \times 10^{-11}$ N m$^2$/kg$^2$

At room temperature $T = 300$ K and $k_B T = 4.1 \times 10^{-21}$ J

**Integrals**

$$\int_{-\infty}^{\infty} dx \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx \ x^n \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n$$

for $n = 2, 4, 6 \cdots$

$$\int_0^{\infty} dx \ x^n \ e^{-x} = n!$$

**Series**

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$1 + x + x^2 + \cdots x^{P-1} = \frac{1 - x^P}{1 - x} \qquad \text{(geometric series)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \text{Zeta function} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \qquad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056 \cdots$$

**Hyperbolic functions**

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

**Stirling's approximation for log of factorial**

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \mathcal{O}(1)) \qquad \ln n! = n\ln n - n + \mathcal{O}(\ln n)$$

**Combinations**

$$C_n^N = \frac{N!}{n!(N-n)!}$$