Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Friday, September 16, 2011, 9 am - 1 pm

Classical and Statistical Mechanics

This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the examination for a total of four solved problems.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Classical Mechanics - solve 2 out of 3 problems

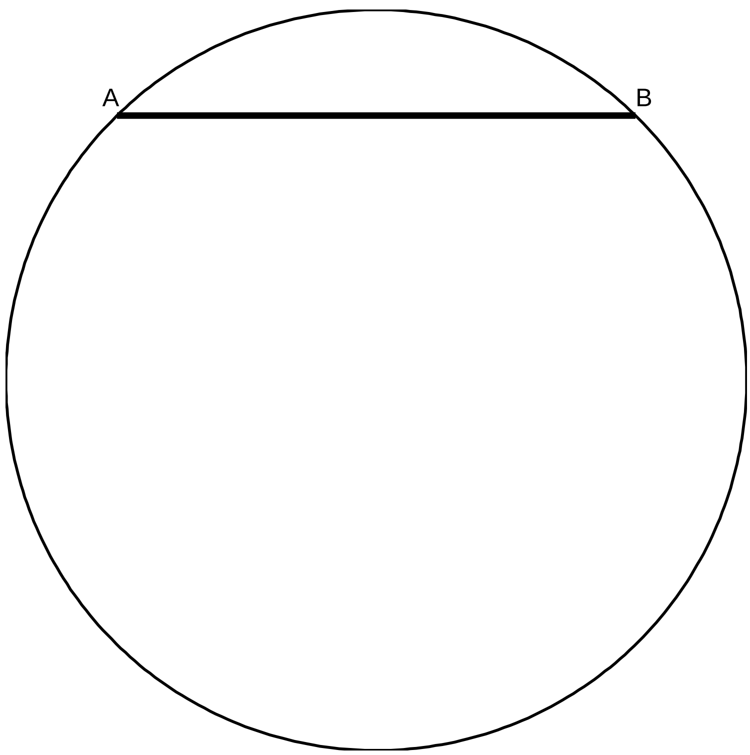
If you solve all 3 problems, only the first 2 will be graded.

1. One way, in principle, to travel between two points on the Earth's surface is simply to dig a tunnel between them. If one ignores friction and other dissipative effects, an object "dropped" from one end of the tunnel will "fall" and eventually surface on the other end, where it can be picked up. This means of transportation uses only the force due to the planet's gravity and, as you will show, can be really fast!

   (a) Assuming that the Earth is a perfect sphere and that its density is constant, show that, for a straight tunnel, the time t_{trip} it takes to "fall" between any two points A and B on the Earth's surface is independent of the positions of A and B on the Earth's surface [see figure].

   (b) For the Earth, estimate t_{trip} in minutes. Remember that the radius of the Earth is approximately 6400 km and that the acceleration due to gravity at the Earth's surface is around 10 m/s^2 [HINT: if you fail to solve part (a), you can still try to answer this part by assuming the premise in (a) is correct.]

   (c) While the straight tunnel is the shortest path between A and B, it is not necessarily the quickest one. Qualitatively sketch the quickest path between two generic points A and B on the Earth's surface. Justify your answer.



2. A point particle of mass m is constrained to move on a surface of revolution characterized by

$$z = \frac{\alpha}{n}\rho^n,$$

$n \neq 0$, where z is the height of the particle, ρ the distance between the particle's position and the z-axis (ρ, ϕ, z are the standard cylindrical coordinates), and α is a positive constant. See the figure for an example. Everywhere the point particle experiences a constant, downwards gravitational force, $\vec{F} = -mg\hat{z}$.

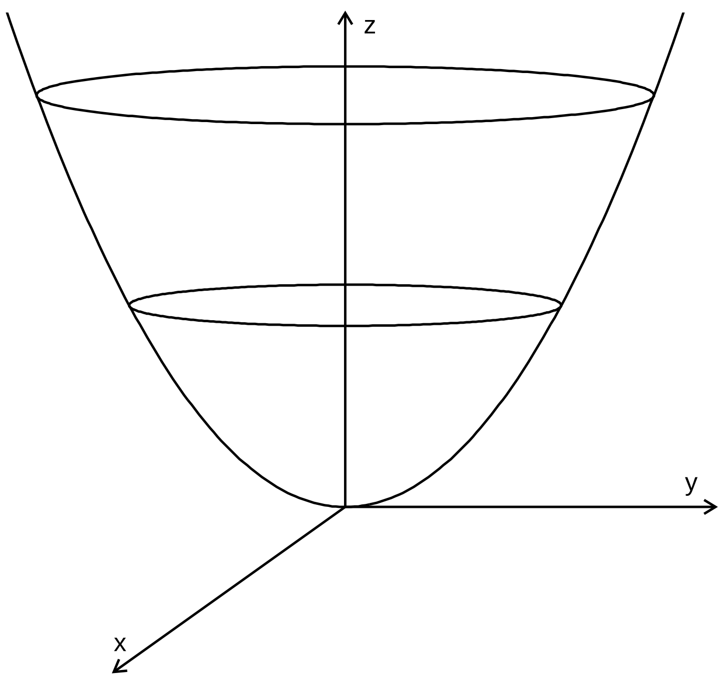
(a) Write down the Lagrangian and the Hamiltonian for the system. Be sure to properly define your coordinates.

(b) Find the equations of motion (Euler-Lagrange equations).

(c) What are the constants of motion for this system?

(d) The initial conditions of the particle are such that it describes a circular orbit $\rho = \rho_0$ perpendicular to the z-axis. What is the orbital period as a function of ρ_0, g, α, m?

(e) For what values of n is the orbit you found in (d) stable?



3. A critically damped one-dimensional harmonic oscillator (point particle, mass m) is characterized by the following equation of motion:

$$m\ddot{x} + 2m\beta\dot{x} + m\beta^2 x = 0.$$

(a) Compute $x(t)$ when $x(0) = x_0$ and $\dot{x}(0) = v_0$. Sketch your result.

(b) The same system, originally at rest at the origin ($x = \dot{x} = 0$ for $t < 0$), is "hit" by an sudden external force, which can be approximated by

$$F(t) = mV\delta(t),$$

where $\delta(t)$ is Dirac's delta-function and V is a constant. The solution to this problem should be familiar – it is the Green's function for the critically damped harmonic oscillator (up to a constant).

(c) The same system is subjected to a time-dependent force

$$F(t) = mA \begin{cases} 0, & t < 0, \\ t/T, & 0 \leq t \leq T, \\ 0, & t > T, \end{cases}$$

where A and T are constants. Compute $x(t)$ assuming that $x(0) = \dot{x}(0) = 0$, for all values of $t \leq T$. Write approximate expressions for $x(t)$, $\dot{x}(t)$ in the limits $t \ll 1/\beta$ and $t \gg 1/\beta$ (but still keeping $t < T$). Comment on your results. [HINT: You may want to take advantage of your solution to (b).]

(d) For the scenario discussed in item (c), compute the total work done on the system by the external force.

(e) For the scenario discussed in item (c), what is the state of the system after a very long time ($t \gg T$ and $t \gg 1/\beta$)? In light of your result to (d), comment on the time dependence of the total mechanical energy.

Statistical Mechanics - solve 2 out of 3 problems

If you solve all 3 problems, only the first 2 will be graded.

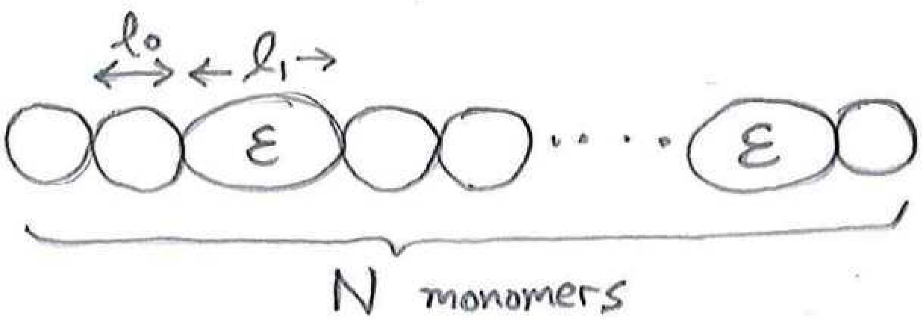
1. Consider a long straight molecule made up of a series of N monomer units, each of which can be in two states. In its low energy state, a monomer is of length ℓ_0, and in the high energy state, it is of length $\ell_1 > \ell_0$. The higher energy state is ϵ above the lower energy state. Each monomer undergoes independent thermal fluctuations between these two states. You may use the thermodynamic limit ($N \gg 1$) to simplify your results.

(a) Find the average length $\langle L \rangle$ of the whole molecule as a function of $k_B T$, ϵ, ℓ_0, ℓ_1 and N.

(b) Find the thermal expansion coefficient of the molecule (fractional change of length with temperature, or $\frac{1}{\langle L \rangle} \frac{\partial \langle L \rangle}{\partial T}$). What is the sign of this quantity and why?

(c) Find the root-mean-squared fluctuation of the molecule length. (note the "root": your result should have dimensions of length) For large N, what is the dependence of the fluctuation on N?

(d) Your results for (a)-(c) are for the molecule with no external forces applied to it. Now suppose a small external force is applied to the molecule, so as to stretch it. Find the *spring constant* of the molecule (the proportionality constant k relating applied force to change in length of the molecule) for small stretching forces. (Hint: the computation can be simplified by using your result for (c))



2. Consider a two-dimensional harmonic oscillator with the following Hamiltonian:

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{k}{2}\left(x_1^2 + x_2^2\right) + \lambda x_1 x_2$$

This oscillator is in thermal equilibrium at temperature T; the spring constant k is positive.

(a) What is the condition on λ for thermal fluctuations of this oscillator to be finite in amplitude?

(b) Find the *classical-mechanical* free energy (by integration over positions and momenta), and the thermal expectation values $\langle x_1^2 \rangle$, $\langle x_2^2 \rangle$ and $\langle x_1 x_2 \rangle$ as a function of λ, in the finite-amplitude regime identified in (a).

(c) Find the *quantum-mechanical* free energy (by summation over quantum states) as a function of λ.

(d) Show that at high temperature the dominant contribution to the free energy of (c) matches the free energy of (b).
Find the characteristic temperature above which the free energies of (b) and (c) are in accord (*i.e.*, the temperature above which the classical description is valid).

3. Consider a classical-mechanical ideal gas containing molecules of mass m. The gas is put into cylindrical tubes of length ℓ which are then put into a centrifuge, at room temperature (300 K). The centrifuge generates a large centrifugal acceleration a along the cylindrical axis of the tubes ($a \gg g$, the gravitational acceleration), which tends to drive the gas towards the bottom of the tubes (towards the outer edge of the centrifuge).

(a) When the gas reaches thermal equilibrium in the centrifuge, find its distribution along the cylindrical axis of the tube up (in the z direction, with $z = 0$ at the bottom of the tube and $z = \ell$ at the top) up to a normalization constant.
Find an expression for the acceleration needed to make a strongly nonuniform distribution.

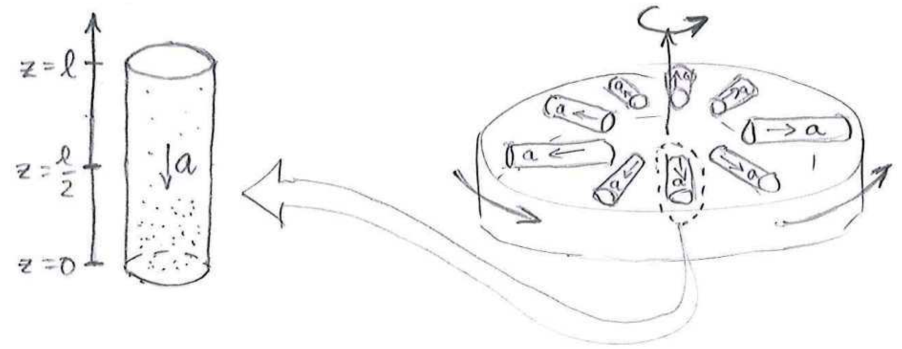
(b) Find the *fraction* f of the molecules of the gas that are in the upper half of the tube (from $z = \ell/2$ to $z = \ell$). What happens to this fraction as the acceleration is increased?

Now suppose a mixture of equal numbers of two species of molecules with similar masses m and M ($M > m$, with $(M - m)/m \ll 1$) are put into the tube. The two molecules will have different distributions along the tube axis. In particular the fraction of the lighter molecules in the upper half of the tube will be greater than the fraction of the heavier molecules in the upper half of the tube, or $f_m > f_M$. So, centrifugation provides a way to *enrich* the relative amount of the lighter species in the upper half of the tube.

(c) Find an *approximate* formula for the acceleration a that corresponds to $f_m = 2f_M$, *i.e.*, to a enrichment by a factor of two of the lighter species in the upper half of the tube. (Hint: use the large-acceleration limit to simplify your formula).

(d) Given m and M are both roughly 10^3 hydrogen atom masses, $\ell \approx 10$ cm, and $M - m \approx$ 10 hydrogen atom masses, estimate what acceleration a is needed to generate enrichment of the lighter species in the upper half of the tube by a factor of two.

(e) Now suppose that the top half of the gas can be extracted from the tube after reaching equilibrium in the spinning centrifuge as in (c). If that (two-fold light-species-enriched) gas is reintroduced into an empty centrifuge tube, it can be centrifuged again (using the same acceleration as in (c)). How many centrifugation steps of this type are needed to reach greater than 99% purity of the lighter species?



Constants

$k_B = 1.381 \times 10^{-23}$ J/K $h = 6.626 \times 10^{-34}$ J·sec $c = 2.998 \times 10^{8}$ m/sec
$m_p = 1.673 \times 10^{-27}$ kg $m_n = 1.675 \times 10^{-27}$ kg $m_e = 9.109 \times 10^{-31}$ kg
$e = 1.602 \times 10^{-19}$ C $G = 6.67 \times 10^{-11}$ N m^2/kg^2

At room temperature $T = 300$ K and $k_B T = 4.1 \times 10^{-21}$ J

Integrals

$$\int_{-\infty}^{\infty} dx\ \exp\left[-\frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx\ x^n\ \exp\left[-\frac{x^2}{2\sigma^2}\right] = 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n$$

for $n = 2, 4, 6 \cdots$

$$\int_0^{\infty} dx\ x^n\ e^{-x} = n!$$

Series

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$1 + x + x^2 + \cdots x^{P-1} = \frac{1 - x^P}{1 - x} \qquad \text{(geometric series)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \text{Zeta function} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \qquad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056\cdots$$

Hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

Stirling's approximation for log of factorial

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \mathcal{O}(1)) \qquad \ln n! = n\ln n - n + \mathcal{O}(\ln n)$$

Combinations

$$C_n^N = \frac{N!}{n!(N-n)!}$$