Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Monday, September 24, 2012, 9 am - 1 pm

Classical and Statistical Mechanics

This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the examination for a total of four solved problems.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Classical Mechanics - solve 2 out of 3 problems
If you solve all 3 problems, only the first 2 will be graded.

1. Two point-like particles, each with charge q and mass m, are confined to move without friction along the circumference of a circle of radius R, as depicted in the figure. A constant electric field $\vec{E} = E\hat{x}$ permeates the entire space, and you are allowed to ignore the effects of any other external forces acting on the system.

   The magnitude of the charges is expressed in units such that the repulsive force between the two particles is

$$\vec{F}_{12} = \frac{q^2}{r^2}\hat{r} = -\vec{F}_{21},$$

   where $\vec{r} = r\hat{r} \equiv \vec{r}_2 - \vec{r}_1$ is the distance vector between the point charges (here labelled '1' and '2'). The electric field translates into a force $\vec{F}_E = q\vec{E}$ on each point charge.

   (a) What is the Lagrangian for this system? Express your answer using the coordinates

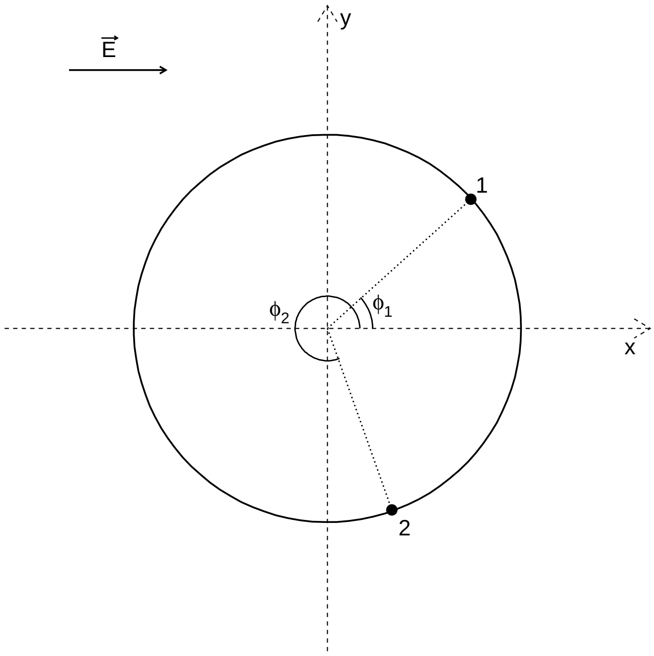
$$\Phi = \frac{1}{2}(\phi_2 + \phi_1) \qquad \text{and} \qquad \varphi = \frac{1}{2}(\phi_2 - \phi_1),$$

   where ϕ_1 or ϕ_2 is the angle that the charge 1 or 2, respectively, defines with respect to the x-axis (see figure).

   (b) What are the equilibrium configurations of this system? Comment on their stability.

   (c) Describe the motion of the system around a stable equilibrium configuration (note: at least one must exist!). Compute all eigenfrequencies and associated eigenmodes.

   (d) Comment on what happens to the equilibrium configurations and the motion around the stable equilibrium point identified in (c) (eigenfrequencies and eigenmodes) when $E = 0$ (no electric field).



Hint – You may find the following well-known trigonometrical fact useful: for a generic triangle with sides a, b, c, the law of cossines states $c^2 = a^2 + b^2 - 2ab\cos\theta_{ab}$, where θ_{ab} is the angle defined by the sides a, b.

2. Two satellites of mass m and $M \geq m$ are placed in circular orbits of identical radius R around the Earth. Since the orbits are coplanar and the satellites are moving in opposite directions, they are destined to collide before completing a single circular orbit. Below, you may assume that the Earth is a perfect sphere and that the satellites are point-like objects.

   (a) If $R = 2\times 10^4$ km, estimate the maximum time it will take the satellites to collide after each has reached its circular orbit. Remember that the radius of the Earth is approximately $R_{\mathrm{E}} = 6.4 \times 10^3$ km and that $g = GM_E/R_E^2 \sim 10$ m/s^2, where M_E is the mass of the Earth and G is Newton's constant.

   (b) Assume that the collision is perfectly inelastic (i.e., the two satellites merge into one after the collision). Describe the trajectory of the system for all values of $m/M \in [0, 1]$. As a function of m/M, how close do the merged satellites get to the center of the Earth? You don't need to check if this is smaller than the Earth's radius R_E.

   (c) Assume that the collision is perfectly elastic. Are there values of m/M for which the lighter satellite is ejected from the Earth's influence?

3. The density of an inhomogeneous rigid cube of side $2s$ and mass M is given by

$$\rho(x,y,z) = \rho_0 \left(1 + \frac{xy + xz + yz}{3s^2}\right),$$

where $x, y, z \in [-s, s]$ are defined in the cube's body reference frame (origin in the center of the cube) and ρ_0 is a constant. There are no external forces acting on the cube. Initially, at $t = 0$, the cube is spinning with initial angular velocity $\vec{\omega}_0 = \omega_0 \hat{z}$ and its body reference frame coincides with the laboratory frame.

(a) Compute ρ_0 as a function of s, and M.

(b) Compute the inertia tensor $\mathbf{I}$ with respect to the origin at $t = 0$. What are the principal moments of this cube with respect to the origin?

(c) Compute, as a function of s, M and $\vec{\omega}_0$, the total initial ($t = 0$) kinetic energy T_0 and angular momentum $\vec{L}_0$ with respect to the origin of the cube.

(d) Are the angular velocity $\vec{\omega}$, angular momentum $\vec{L}$ with respect to the origin, and kinetic energy T of the cube expected to remain constant for $t > 0$? (i.e., is $\vec{\omega} = \vec{\omega}_0$, $\vec{L} = \vec{L}_0$, and $T = T_0$ for all $t > 0$?) Justify your answer.

(e) Assume now that the cube is in the same initial configuration but $\vec{\omega}_0 = \omega_0 \frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}$. Answer (d) under these circumstances.

Recall that the components of the inertia tensor are

$$\mathbf{I}_{ij} = \int \mathrm{d}^3 r \rho(\vec{r}) \left(\delta_{ij} \sum_{k=1}^{3} r_k^2 - r_i r_j\right),$$

where $r_1 = x$, $r_2 = y$, $r_3 = z$, and $\mathrm{d}^3 r = \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

Statistical Mechanics - solve 2 out of 3 problems

If you solve all 3 problems, only the first 2 will be graded.

1. (a) A spin-S spin with total spin quantum number S is in a magnetic field. In terms of the spin operator $\mathbf{S}$ the Hamiltonian is

$$H = -\mu \frac{\mathbf{S}}{\hbar} \cdot \mathbf{B}$$

where $\mathbf{B}$ is the constant magnetic field and μ is its magnetic moment. Write down the partition function for the spin in the field and at temperature T (your result will be a finite sum).

(b) Find an expression for the average energy of the spin
(*i.e.*, the expectation value $\langle H \rangle$; your result will be a ratio of finite sums).

(c) Now, consider a "classical" spin, a freely rotating magnetic moment described by a unit vector $\mathbf{n}$ that can point in any direction in 3d space (the vector $\mathbf{n}$ is on the unit sphere). The Hamiltonian of this spin is

$$H_c = -\mu_c \hat{\mathbf{n}} \cdot \mathbf{B}$$

Compute its partition function by integrating over all orientations of $\hat{\mathbf{n}}$.

(d) Show that for sufficiently small magnetic fields or for sufficiently large temperature, your result of (a) approaches the limit of (c) in the $S \to \infty$ limit (up to a constant factor in Z which is just a shift in the free energy).
Find the relationship between μ and μ_c in that limit.

This indicates why we sometimes talk about the large-S limit as a "classical" limit.

2. Consider a *non-ideal* but dilute gas of N identical, zero-spin, nonrelativistic particles with Hamiltonian

$$H = \frac{1}{2m}\sum_{i=1}^{N}\mathbf{p}_i^2 + \sum_{i=1}^{N}\sum_{j=i+1}^{N} U(|\mathbf{r}_i - \mathbf{r}_j|)$$

confined to a cubic box of side L (volume $V = L^3$). Note that there are interactions between the particles in the form of a central potential $U(r)$ acting between each distinct pair of particles. The range of the potential $U(r)$ (distance over which it decays to much less than $k_B T$) is much smaller than L, and smaller than the typical distance between the particles.

(a) What is the condition on temperature, particle mass m, and size of the container L such effects of quantization of momentum states will be negligible (you may ignore effects of the potential U).

(b) Write down an expression for the partition function $Z(N, V, T)$ for this gas, the position and momentum degrees of freedom *classically*.
(*i.e.*, the temperature is high enought that effects of quantum statistics and momentum quantization may be neglected)
You should integrate over the momentum degrees of freedom but **not** the position degrees of freedom; your result will still have the position integrals in it.

The following three parts of this problem concern the system discussed above, and can be expressed in terms of the partition function $Z(N, T, V)$ of (b), without actually computing it.

(c) Find expressions for the internal energy (the expectation value of the Hamiltonian, $E = \langle H \rangle$) and the pressure p of the gas in terms of Z.

(d) Show that $(\partial p/\partial T)_{N,V} \propto (\partial S/\partial V)_{N,T}$ where S is the entropy, and determine the proportionality constant.

(e) Can $[(\partial E/\partial T)_{N,T}]$ ever be negative? Use Z to show why or why not.

3. N identical particles each of mass m are constrained to move in one dimension (x). Each particle is subject to a one dimensional harmonic oscillator potential well ($U(x) = kx^2/2$). There are no interactions between the particles, but their quantum statistics are relevant.

(a) Suppose the particles are spin-3/2. At zero temperature, find the energy $E_{\max}$ of the highest occupied quantum state. What is the usual name of this characteristic energy in the theory of quantum gases?

(b) What is the total energy of the system of N spin-3/2 particles of (a) (still at zero temperature) in terms of $E_{\max}$?

(c) Suppose the particles of parts (a)-(b) are replaced by N noninteracting identical spin-1 particles, with temperature T still held at zero. What is the highest occupied quantum state?

(d) For (c), suppose the temperature is slowly raised up from zero. At what characteristic (approximate) temperature do you start to see excitations above the ground state, and what is the energy of the most probable excited state just as excitations start to occur?

Constants

$k_B = 1.381 \times 10^{-23}$ J/K $\quad h = 6.626 \times 10^{-34}$ J·sec $\quad c = 2.998 \times 10^{8}$ m/sec
$m_p = 1.673 \times 10^{-27}$ kg $\quad m_n = 1.675 \times 10^{-27}$ kg $\quad m_e = 9.109 \times 10^{-31}$ kg
$e = 1.602 \times 10^{-19}$ C $\quad G = 6.67 \times 10^{-11}$ N m^2/kg^2

At room temperature $T = 300$ K and $k_B T = 4.1 \times 10^{-21}$ J

Integrals

$$\int_{-\infty}^{\infty} dx \; \exp\left[-\frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx \; x^n \; \exp\left[-\frac{x^2}{2\sigma^2}\right] = 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n$$

for $n = 2, 4, 6 \cdots$

$$\int_0^{\infty} dx \; x^n \; e^{-x} = n!$$

Series

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$1 + x + x^2 + \cdots x^{P-1} = \frac{1 - x^P}{1 - x} \qquad \text{(geometric series)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \text{Zeta function} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \qquad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056 \cdots$$

Hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

Stirling's approximation for log of factorial

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \mathcal{O}(1)) \qquad \ln n! = n\ln n - n + \mathcal{O}(\ln n)$$

Combinations

$$C_n^N = \frac{N!}{n!(N-n)!}$$