Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Wednesday, September 16, 2015, 9 am - 1 pm

Classical and Statistical Mechanics

This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the examination for a total of four solved problems.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Classical Mechanics - solve 2 out of 3 problems

If you solve all 3 problems, only the first 2 will be graded.

1. a) A rod of length $2L$ slides on a frictionless table. Its initial velocity has magnitude v_i, and is perpendicular to its axis. It then collides elastically with a peg that is fixed in the table at a distance of $f \cdot L$ from the rod's center (where $0 \leq f \leq 1$). Use the appropriate conservation laws to determine the velocity of the center of mass of the rod after the collision.

b) Repeat 1a, but this time calculate the center of mass velocity immediately after the collision when the collision is fully inelastic, i.e., assume that the rod sticks to the peg (with the peg pivoting freely).

2. Consider two bodies of mass m_1 and m_2 that are on circular orbits about their common center of mass. Their orbital frequency is Ω. Body m_2 is a point mass. Body m_1 has a moment of inertia I due to its spin, and it spins at frequency ω (in the same sense as the orbital motion, i.e., its spin vector is aligned with the angular momentum vector of the orbital motion). Do *not* assume that one of the bodies is much less massive than the other.

a) Derive their orbital separation a in terms of m_1, m_2, Ω and the gravitational constant G.

b) Derive the total energy.

c) Derive the total angular momentum.

d) For fixed angular momentum, show that the energy is minimized at synchronism, i.e. when $\omega = \Omega$.

3. Two masses are connected by two springs. The masses and springs are forced to move along a circle. In equilibrium they look like this:

The radius of the circle is R, the spring constants of both springs are k, and the masses are m_1 and m_2. Choose the generalized co-ordinates to be the angular displacements of the two masses from equilibrium (θ_1 and θ_2).

a) Determine the Lagrangian.

b) Find the eigenfrequencies.

c) Describe qualitatively the two normal modes.

Statistical Mechanics - solve 2 out of 3 problems

If you solve all 3 problems, only the first 2 will be graded.

1. Consider an ideal gas of identical spinless atoms in volume V, at temperature T, at chemical potential μ (the number of atoms in the gas is not fixed, so the gas density is controlled by μ). Each atom is of mass m.

(a) What two conditions are required in order to treat the atoms in the volume V as "classical" particles (with their position and momentum states enumerated independently and classically, and with no dependence of their thermodynamics on their statistics)? You should express these conditions in terms of the average particle number $\langle N \rangle$. Assume that this/these condition(s) is/are satisfied for the remainder of this problem.

(b) Compute the average number density of the gas (a function of T, μ, m, h, k_B).

(c) Find a condition for the chemical potential, such that the number of atoms in the volume V is much larger than 1.

(d) Compute the internal energy of the gas as a function of μ and V, and then finally express your result in terms of the average particle number $\langle N \rangle$.

2. N electrons are confined to a two-dimensional square of area A so as to form an ideal quantum gas (you may ignore Coulomb interactions between the electrons).

(a) How large does A have to be in order to accurately enumerate the quantum states of the electrons using a momentum integral?

Assume that condition (a) holds for the remainder of the problem.

(b) Find the internal energy of the 2D electron gas at $T = 0$.

(c) Still considering $T = 0$, suppose we open a gate so that the gas can access a second square 2D region also of area A, but at an energy ϵ above that of the 2D region considered in (a)-(c). As N is increased, at what N does this second region start to affect the result of (b)?

(d) What is the characteristic temperature T above which the internal energy differs appreciably from the $T = 0$ result of (b)?

3. Consider a crystal containing N spin-1 spins, of magnetic moment μ, in thermal equilibrium at temperature T and external magnetic field $H\hat{\mathbf{z}}$. There are no interactions between the spins, but each spin is coupled to the external magnetic field.

(a) Write the Hamiltonian suitable to describe the N spins interacting with the external field.

(b) Find the free energy of the spins as a function of N, H and T. What dimensionless combination of H and T is the free energy a function of?

(c) Find the magnetization of the spins as a function of N, H and T. Sketch a graph of the magnetization as a function of the dimensionless combination of H and T found in (b).

(d) Find the specific heat of the spins as a function of N, H and T. At what approximate temperature do you find a peak in the specific heat? Sketch the specific heat as a function of the dimensionless combination of H and T found in (b).

(e) Derive a relation between the specific heat of (d) and the mean-squared fluctuation of the total magnetization.

Constants

$k_B = 1.381 \times 10^{-23}$ J/K $\quad h = 6.626 \times 10^{-34}$ J·sec $\quad c = 2.998 \times 10^{8}$ m/sec
$m_p = 1.673 \times 10^{-27}$ kg $\quad m_n = 1.675 \times 10^{-27}$ kg $\quad m_e = 9.109 \times 10^{-31}$ kg
$e = 1.602 \times 10^{-19}$ C $\quad G = 6.67 \times 10^{-11}$ N m^2/kg^2

At room temperature $T = 300$ K and $k_B T = 4.1 \times 10^{-21}$ J

Integrals

$$\int_{-\infty}^{\infty} dx \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx \ x^n \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n$$

for $n = 2, 4, 6 \cdots$

$$\int_{0}^{\infty} dx \ x^n \ e^{-x} = n!$$

Series

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$1 + x + x^2 + \cdots x^{P-1} = \frac{1 - x^P}{1 - x} \qquad \text{(geometric series)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \text{Zeta function} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \qquad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056 \cdots$$

Hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

Stirling's approximation for log of factorial

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \mathcal{O}(1)) \qquad \ln n! = n\ln n - n + \mathcal{O}(\ln n)$$

Combinations

$$C_n^N = \frac{N!}{n!(N-n)!}$$