**Northwestern University Physics and Astronomy Ph.D. Qualifying Examination**

Friday, June 14, 2013, 9 am - 1 pm

**Classical and Statistical Mechanics**

**This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the examination for a total of four solved problems.**

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

**Classical Mechanics - solve 2 out of 3 problems**

If you solve all 3 problems, only the first 2 will be graded.

## Problem 1

A uniform, circular ring of mass $M$ and radius $r$ spins freely about a vertical spindle running through the diameter of the ring, as shown below. A bead of mass $m$ slides freely along the circumference of the ring. There is a constant downward gravitational field $g$.

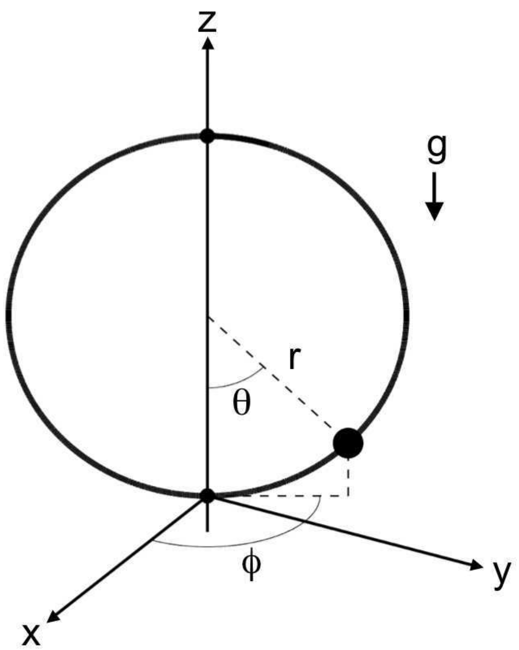
(a) Calculate the moment of inertia of the ring (without the bead) about the spindle.

(b) Write the Lagrangian for this system (including the bead). You may use the coordinates $\theta$ and $\phi$ as shown below.

(c) Find the two equations of motion for this system.

(d) One of the equations of motion from (c) gives a conserved quantity. Write an expression for this conserved quantity and use it to eliminate the azimuthal angle ($\phi$ and $\dot{\phi}$ if you used the coordinates shown below) from the other equation of motion.

(e) Show from (d) that there is an equilibrium point at the bottom of the ring. Find the frequency of small oscillations about this point, in terms of the conserved quantity from (d). Is this equilibrium point always stable? If not, give the condition for stability.



## Problem 2

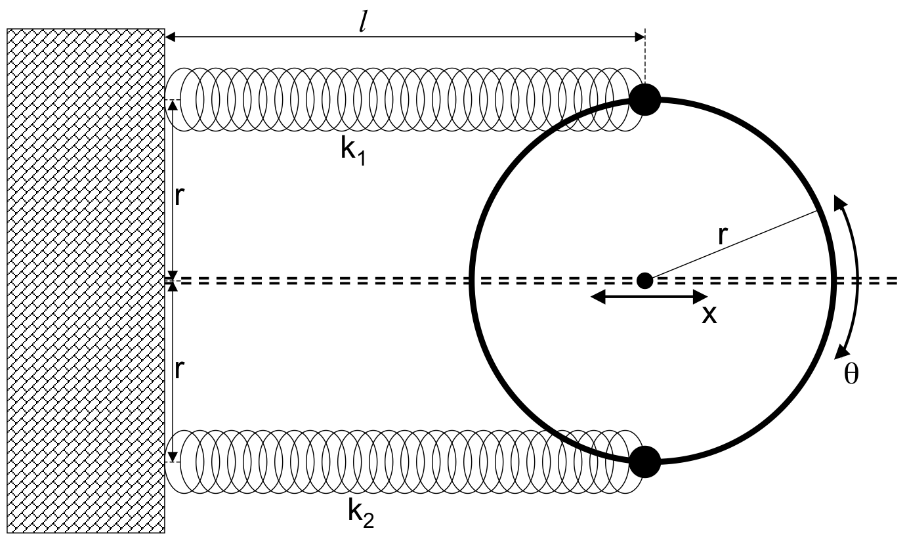
This problem investigates small oscillations of the hoop in the 2-D system shown below. The hoop is circular with radius $r$ and uniform mass $m$. The hoop is constrained to slide along a track and to rotate in the plane of the figure about its (the hoop's) center. These two motions are indicated by $x$ and $\theta$ in the figure. Two springs, with spring constants $k_1$ and $k_2$ and equilibrium length $\ell$, are attached at opposite points on the hoop, connecting it to a wall. In the equilibrium position the springs are parallel to the track and perpendicular to the wall.

(a) Consider the case where $k_1 = k_2 = k$. By inspection, what do you expect the normal modes of small oscillations to be? Show that these are in fact normal modes for the system and find their frequencies.

(b) Now consider general $k_1$ and $k_2$. Find the eigenfrequencies for the normal modes of the system.

(c) Find the eigenvectors for the eigenfrequencies from (b). Qualitatively describe the motion of these modes.

(d) Consider the modes described in (c) for the case $k_1 = k_2 = k$. Are these the same modes you described in (a)? If not, explain how this is consistent.



## Problem 3

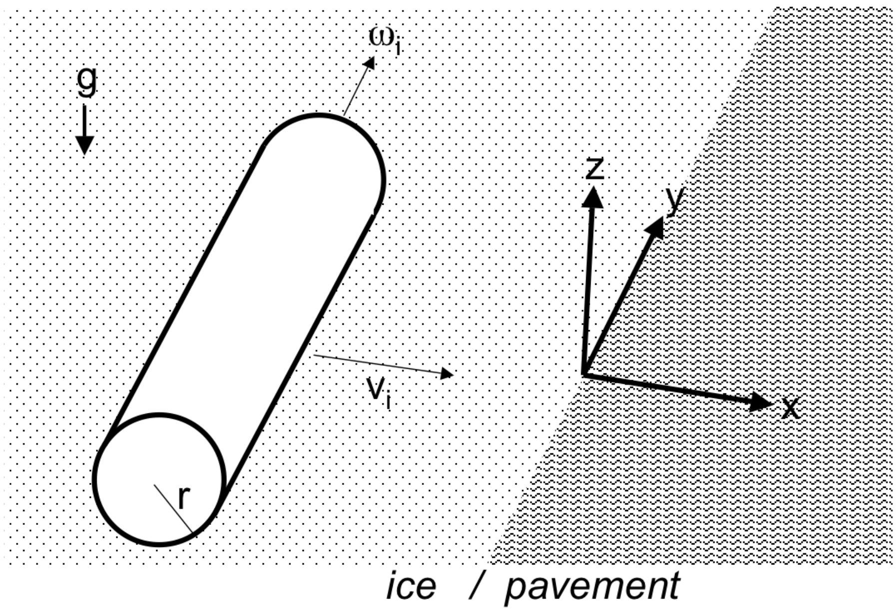
A *hollow* cylinder of mass $m$ and radius $r$ slides on a level, icy (frictionless) surface in the $+x$ direction with initial velocity $v_i$, as shown below. The cylinder's axis is in the $y$ direction, and the cylinder initially has no angular velocity ($\omega_i = 0$). At $x = 0$, the surface changes to pavement (friction becomes large), so that the cylinder quickly begins rolling without slipping.

(a) Write down the condition for rolling without slipping in terms of the linear velocity $v$ and angular velocity $\omega$.

(b) Draw a diagram of the forces at work as the cylinder transitions from sliding to rolling. Find a relation between the linear and angular acceleration of the cylinder.

(c) Determine the final linear and angular velocity of the cylinder.

(d) Suppose the cylinder had an initial "backwards" angular velocity ($\omega_i < 0$ in the figure below). How large does this angular velocity need to be to give a final linear velocity in the $-x$ direction?



**Statistical Mechanics - solve 2 out of 3 problems**

If you solve all 3 problems, only the first 2 will be graded.

**1.** (a) Consider a molecule which has an energy spectrum $E_n = n\epsilon_0$, for $n = 0, 1, 2, \cdots$, with one state per energy level. Compute the partition function for this molecule at temperature $T$ (you may ignore degrees of freedom other than $n$, e.g., translation and rotation).

(b) Compute the average energy and the specific heat for the molecule and sketch them as a function of temperature.

(c) Suppose that the molecule now has $2^n$ degenerate states in energy level $n$. Compute the partition function for the molecule, and find the entropy as a function of temperature.

(d) Your result for (c) will show a divergence in the partition function as one approaches a particular "critical temperature". Find that critical temperature and explain what is going on as it is approached from below.

**2.** Suppose we have a particle undergoing a random walk on the $x$-axis. At time $t = 0$, $x = 0$. For times of $t = \tau$, $t = 2\tau$, and so on, the particle moves either to the left or the right by an amount $\Delta$. At each time step the probability of stepping to the right is $r$, and to the left is $1 - r$. The successive steps are independent of one another.

We repeat observations of enough particles of this type that we can form averages of $x(t)$, $x^2(t)$ and so on (the particles in the different observations are completely independent of one another). We take all particles to have $x = 0$ at $t = 0$.

(a) Find the average displacement $\langle x \rangle$ after $N$ steps (at time $N\tau$). What is the average velocity of the randomly walking particles?

(b) Find the variance of the displacement, $\left\langle (x - \langle x \rangle)^2 \right\rangle$.

(c) Find the probability distribution for $x$ as a function of $r$ and $N$ [exact formula for $P(x, N, r)$].

(d) In the limit $N \to \infty$ find the limiting form of the distribution of $x$ and sketch for the cases $r = 1/2$, $r = 3/4$ and $r = 1$. Mark and label your sketches informatively.

Note that (d) can be solved without solving (c).

**3.** (a) For an ideal Fermi gas of $N$ spin-1/2 particles in volume $V$ in three dimensions with an energy per particle of $p^2/2m$, find the Fermi energy, the ground-state energy, and the pressure at zero temperature.

(b) Use your results of (a) to *estimate* the Fermi energy and the Fermi velocity for the conduction electrons in copper. The lattice constant for copper is 3.6 Å(don't worry too much about the details of the lattice - your estimate will be approximate).

(c) Suppose that an ideal Fermi gas of $N$ spin-1/2 particles is confined to a two-dimensional area $A$, and that the energy per particle is of $\alpha p$ for some constant $\alpha$. As in (a), find the Fermi energy, the ground-state energy, and the two-dimensional pressure at zero temperature.

**Constants**

$k_B = 1.381 \times 10^{-23}$ J/K $\quad h = 6.626 \times 10^{-34}$ J·sec $\quad c = 2.998 \times 10^{8}$ m/sec
$m_p = 1.673 \times 10^{-27}$ kg $\quad m_n = 1.675 \times 10^{-27}$ kg $\quad m_e = 9.109 \times 10^{-31}$ kg
$e = 1.602 \times 10^{-19}$ C $\quad G = 6.67 \times 10^{-11}$ N m$^2$/kg$^2$

At room temperature $T = 300$ K and $k_B T = 4.1 \times 10^{-21}$ J

**Integrals**

$$\int_{-\infty}^{\infty} dx \;\exp\left[-\frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx\; x^n \;\exp\left[-\frac{x^2}{2\sigma^2}\right] = 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n$$

for $n = 2, 4, 6 \cdots$

$$\int_0^{\infty} dx\; x^n\; e^{-x} = n!$$

**Series**

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$1 + x + x^2 + \cdots x^{P-1} = \frac{1 - x^P}{1 - x} \qquad \text{(geometric series)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \text{Zeta function} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \qquad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056\cdots$$

**Hyperbolic functions**

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

**Stirling's approximation for log of factorial**

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \mathcal{O}(1)) \qquad \ln n! = n\ln n - n + \mathcal{O}(\ln n)$$

**Combinations**

$$C_n^N = \frac{N!}{n!(N-n)!}$$