Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Friday, September 19, 2014, 9 am - 1 pm

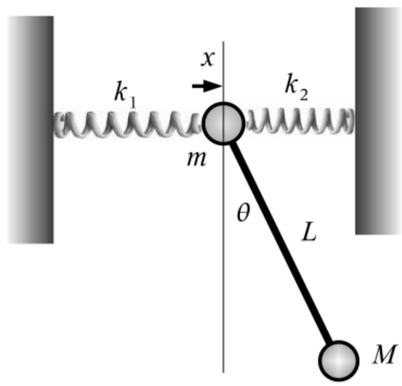
Classical and Statistical Mechanics

This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the examination for a total of four solved problems.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Classical Mechanics - solve 2 out of 3 problems
If you solve all 3 problems, only the first 2 will be graded.

1. Consider the system below. The ball of mass m can move only horizontally. It is attached via a massless rod to a second ball of mass M. Gravity acts downwards (gravitational acceleration is g).



(a) What is the Lagrangian $\mathcal{L}(x, \dot{x}, \theta, \dot{\theta})$? (Take x to be the displacement from equilibrium.)

(b) What are the equations of motion?

(c) What are the frequencies of the two normal modes? For this part, assume the amplitude of oscillation is small, i.e. you may linearize the equations in θ and x.

(d) Now set $m \to 0$. What is the frequency of the low frequency mode?

(e) Explain your result from part (d) physically in the limit that the springs are very stiff.

2. A particle of negligibly small mass moves in the gravitational field that is caused by various distributions of mass. For each of the following distributions, list all conserved quantities in the motion of the particle. Express your answers in terms of the particle's co-oordinates (x, y, z) and velocities $(\dot{x}, \dot{y}, \dot{z})$, as well as any other parameters of the problem (but define those clearly).

a) The mass is uniformly distributed in the plane $z = 0$.

b) The mass is uniformly distributed in the half-plane $z = 0$, $y > 0$.

c) The mass is uniformly distributed in a circular cylinder of infinite length, with axis along the z-axis

d) The mass is uniformly distributed in a circular cylinder of finite length, with axis along the z-axis

e) The mass is uniformly distributed in a "right elliptical cylinder" (i.e., a cylinder that has an elliptical cross section) of infinite length, with axis along the z-axis

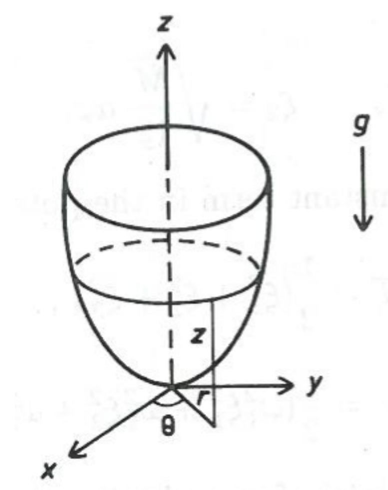
f) The mass is uniformly distributed in a dumbbell whose axis is oriented along the z-axis.

g) The mass is in the form of a uniform wire wound in the geometry of an infinite helical solenoid, with axis along the z axis.

3: A particle moves on the inside wall of a bowl whose height is given by

$$z = \frac{kr^2}{2} \tag{1}$$

where r is the cylindrical radius ($r = \sqrt{x^2 + y^2}$). The gravitational acceleration is g.



a) What is the Lagrangian and what are the equations of motion? Use r and θ as your co-ordinates.

b) The particle is on a circular orbit at height $z = z_0$. What is its orbital angular frequency?

c) The particle from part (b) is given a tiny kick downwards. Use angular momentum conservation to determine $d\theta/dt = f(r)$ after the kick; i.e., determine the function $f(r)$. That function should also depend on the parameters g, k, and z_0.

d) What is the linearized equation of motion for r for the kicked particle described in part (c)? You should use your result from part (c) to eliminate θ from the equation of motion, and assume that the deviation from the pre-kick orbit is very small (so that you may linearize the equation of motion).

e) What is the frequency of oscillation of r after the kick, in terms of g, k, and z_0?

Statistical Mechanics - solve 2 out of 3 problems

If you solve all 3 problems, only the first 2 will be graded.

1. Consider an ideal gas of $N \gg 1$ identical particles in a box of volume V, at temperature T. Each particle is of mass m.

(a) Under what circumstances can the gas be treated as "classical", with no dependence of the properties of the gas on the quantum statistics of the particles?

(b) Find the canonical partition function for the gas in the "classical" limit of (a).

(c) Find the pressure and the total entropy of the gas (you may ignore quantum statistics of the particles).

(d) Suppose we have two containers of gas which have identical V, N, and T, but the particles in one box have twice the mass of the other. Now a pipe between the two containers is opened to allow the two gases to mix in the volume $2V$. Find the change in the pressure and the entropy of the gas.

2. Consider a model of a magnetic crystal composed of N identical atoms, where each atom has a spin-1/2 magnetic moment of magnitude μ, plus three vibrational degrees of freedom of oscillator frequency ω (this vibrational model is often called the Einstein model).

You should consider the spin and vibration degrees of freedom to be in thermal contact (i.e., to be able to exchange energy and to reach equilibrium with one another).

(a) Find the canonical partition function for the spins, and calculate the total magnetic moment of the crystal as a function of magnetic field and temperature.

(b) Use (a) to express the condition on the magnetic field for the spins to be strongly polarized.

(c) Find the canonical partition function for the vibrational degrees of freedom and compute the internal energy as a function of temperature, finding the low-temperature limiting behavior. Below what characteristic temperature do you expect to see your low-temperature limit?

(d) Find the heat capacities of the magnetic and vibrational degrees of freedom at nonzero magnetic field separately, at temperatures where the vibrational degrees of freedom are in the low-temperature limit identified in (c).
Estimate the value of magnetic field at which these two low-temperature heat capacities are equal.

3. In the science section of the *New York Times* it is reported that a new stable, spin-0, massive particle has been discovered with a new type of quantum statistics. Up to *three* of these particles can occupy each quantum state.

(a) For an ideal, nonrelativistic gas of these particles in three dimensions, with one-particle energies $\epsilon(\mathbf{p}) = |\mathbf{p}|^2/(2m)$, find the grand canonical partition function as a function of $k_B T$, chemical potential μ and volume V. You may leave your result in terms of a momentum integral, and you may assume $V \gg \left(mk_BT/h^2\right)^{3/2}$.

(b) Find $\langle N \rangle$, again leaving it as an integral over momentum.

(c) At low temperatures and fixed particle number, do you expect this gas to behave more like fermions (is there a Fermi surface and if so, what is the Fermi energy?), or like bosons (is there Bose condensation and if so, what is T_c?).

Constants

$k_B = 1.381 \times 10^{-23}$ J/K $\quad h = 6.626 \times 10^{-34}$ J·sec $\quad c = 2.998 \times 10^{8}$ m/sec
$m_p = 1.673 \times 10^{-27}$ kg $\quad m_n = 1.675 \times 10^{-27}$ kg $\quad m_e = 9.109 \times 10^{-31}$ kg
$e = 1.602 \times 10^{-19}$ C $\quad G = 6.67 \times 10^{-11}$ N m^2/kg^2

At room temperature $T = 300$ K and $k_BT = 4.1 \times 10^{-21}$ J

Integrals

$$\int_{-\infty}^{\infty} dx \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx \ x^n \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n$$

for $n = 2, 4, 6 \cdots$

$$\int_0^{\infty} dx \ x^n \ e^{-x} = n!$$

Series

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$1 + x + x^2 + \cdots x^{P-1} = \frac{1 - x^P}{1 - x} \qquad \text{(geometric series)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \text{Zeta function} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \qquad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056 \cdots$$

Hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

Stirling's approximation for log of factorial

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \mathcal{O}(1)) \qquad \ln n! = n\ln n - n + \mathcal{O}(\ln n)$$

Combinations

$$C_n^N = \frac{N!}{n!(N-n)!}$$