Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Wednesday, June 11, 2014, 9 am - 1 pm

Classical and Statistical Mechanics

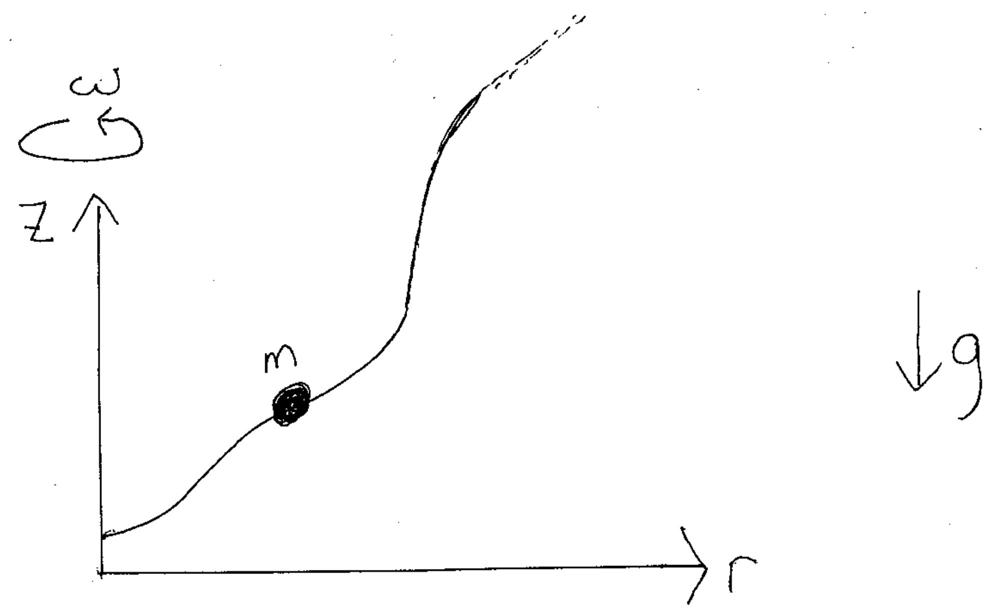
This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the examination for a total of four solved problems.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Classical Mechanics - solve 2 out of 3 problems

If you solve all 3 problems, only the first 2 will be graded.

1. Consider a wire that lies in a 2-D plane, of functional form $z = f(r)$. A bead of mass m slides frictionlessly along the wire, with gravity acting downwards. The wire rotates about the vertical at constant angular frequency ω.



(a) What is the Lagrangian $L(r, \dot{r}, t)$?

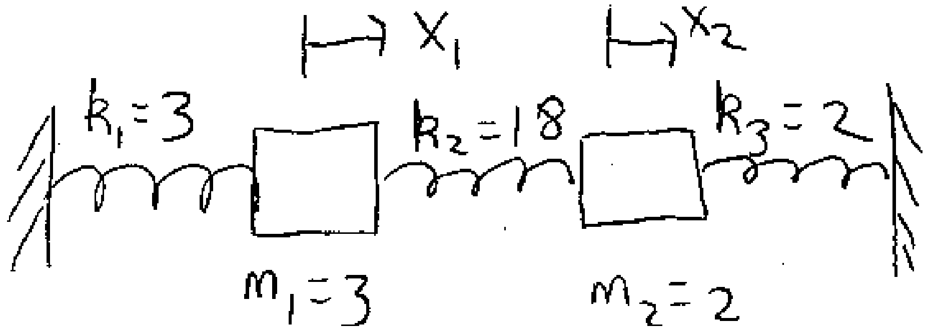
(b) What is the equation of motion for $r(t)$?

(c) Derive an expression for the value of r at which r =const. (Call it r_0).

(d) Derive a condition on the function $f(r)$ so that the solution $r = r_0$ is stable.

(e) Find the Hamiltonian $H(r, p_r, t)$. Is it conserved? Why or why not?

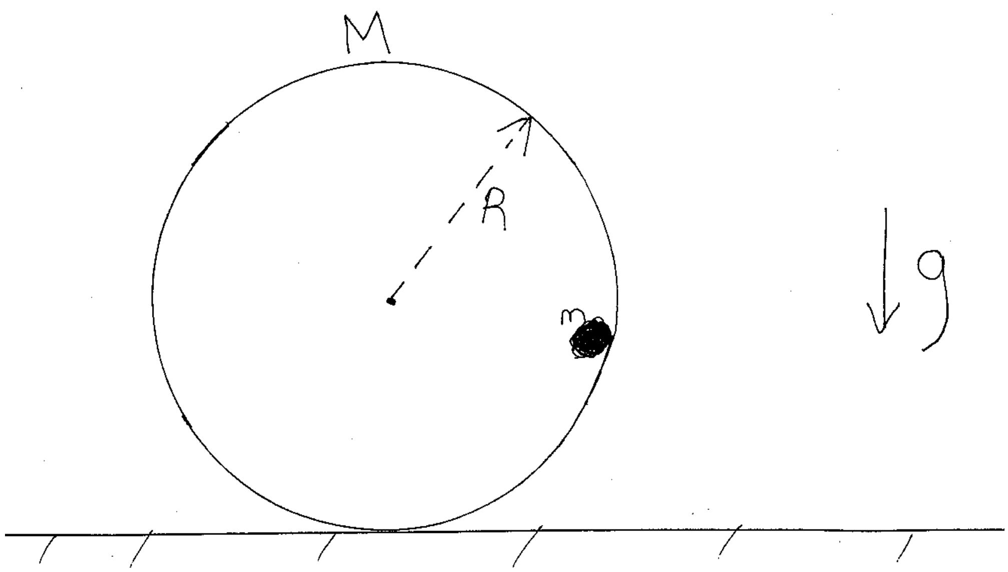
2. (a) Find the normal modes and frequencies for the following system.



(b) Initially, $x_1 = x_2 = 0$ and $v_1 = 1$ and $v_2 = 0$. What are $x_1(t)$ and $x_2(t)$?

(c) How long until the system returns to its initial state?

3. Consider a bead of mass m that slides without friction inside a ring of mass M. The ring rolls without slipping on a horizontal surface. All motion is confined to a vertical plane and gravity acts downwards.



(a) What is the Lagrangian?

(b) What is the frequency of small oscillations about the point of stable equilibrium?

(c) **For this part, do not assume small oscillations.** Initially, the ring and bead are held stationary, with the bead at its rightmost position relative to the ring (at height R above the surface). After the ring and bead are released, what is the translational speed of the ring when the bead is at its lowest position?

Statistical Mechanics - solve 2 out of 3 problems

If you solve all 3 problems, only the first 2 will be graded.

Problem 1.

A $^7N_{14}$ nucleus has nuclear spin +1. Assume that the diatomic N_2 molecule can rotate but not vibrate, and ignore the electrons. Thus the molecule is just two nuclei connected by a bond of fixed length, and has moment of inertia I.

(a) Determine a general expression for N_{ortho}/N_{para}, the relative abundance of ortho-nitrogen (symmetric spin states) and para-nitrogen (antisymmetric spin states) in ideal nitrogen gas at temperature T. Your answer may contain unevaluated sums, integrals, etc.

(b) Evaluate this expression in the two limits $T \to 0$ and $T \to \infty$ (i.e. obtain simple forms for the relative abundance in these two cases).

(c) Define a suitable characteristic temperature such that the low-temperature limit applies for $T \ll T_{char}$ and the high-temperature limit for $T \gg T_{char}$.

Problem 2.

Consider a black box of volume V containing electron-positron pairs and photons in equilibrium at temperature T. Equilibrium is established by reactions such as $e^+ + e^- \leftrightarrow 2\gamma$. Assume that no other pairs are produced.

(a) Find the chemical potentials of the electrons and positrons, assuming that they are present in equal numbers.

(b) Find the average number of electron-positron pairs, in the two limits $kT \gg mc^2$ and $kT \ll mc^2$.

(c) Next assume that the material of the walls serves as a reservoir of electrons but not positrons, so that the numbers of electrons and positrons are not the same. Therefore the electrons in the gas have a positive chemical potential equaling the chemical potential of the electrons in the reservoir, $+\mu_o$. Determine the chemical potential of the positron gas, and calculate the net charge of the gas in the limit $kT \gg \mu_o \gg mc^2$..

NOTE: Answers may be left in terms of dimensionless definite integrals.

Problem 3.

(a) Consider an ensemble of independent one-dimensional simple harmonic *quantum-mechanical* oscillators, in thermal equilibrium with a reservoir at temperature T where $kT \ll \hbar\omega$. Find the mean energy per oscillator.

(b) Next consider an ensemble of anisotropic two-dimensional quantum-mechanical oscillators, with frequencies ω_x and ω_y. What is the partition function in this case, and what is the mean energy per oscillator?

(c) Returning to the one-dimensional case, consider the oscillators to be *classical*, with potential energy $V = cx^2 + gx^4$ where g is small and positive. To first order in g, calculate the heat capacity per oscillator.

Constants

$k_B = 1.381 \times 10^{-23}$ J/K $\quad h = 6.626 \times 10^{-34}$ J·sec $\quad c = 2.998 \times 10^{8}$ m/sec

$m_p = 1.673 \times 10^{-27}$ kg $\quad m_n = 1.675 \times 10^{-27}$ kg $\quad m_e = 9.109 \times 10^{-31}$ kg

$e = 1.602 \times 10^{-19}$ C $\quad G = 6.67 \times 10^{-11}$ N m^2/kg^2

At room temperature $T = 300$ K and $k_B T = 4.1 \times 10^{-21}$ J

Integrals

$$\int_{-\infty}^{\infty} dx\ \exp\left[-\frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx\ x^n\ \exp\left[-\frac{x^2}{2\sigma^2}\right] = 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n$$

for $n = 2, 4, 6 \cdots$

$$\int_{0}^{\infty} dx\ x^n\ e^{-x} = n!$$

Series

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$1 + x + x^2 + \cdots x^{P-1} = \frac{1 - x^P}{1 - x} \qquad \text{(geometric series)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \text{Zeta function} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \qquad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056 \cdots$$

Hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

Stirling's approximation for log of factorial

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \mathcal{O}(1)) \qquad \ln n! = n\ln n - n + \mathcal{O}(\ln n)$$

Combinations

$$C_n^N = \frac{N!}{n!(N-n)!}$$