Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Monday, September 15, 2014, 9 am - 1 pm

Electricity and Magnetism

Solve 3 out of 4 problems

If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Note that you may use the Gaussian or SI systems of units for any problem, but please use the same system throughout the entirety of each problem.

1. In an approximate model of the Helium atom, the charge densities of the two electrons are given by

$$\rho_1(\mathbf{r}) = \rho_2(\mathbf{r}) = \frac{e}{\pi a^3} e^{-2r/a},$$

where the subscripts 1 and 2 refer to the two electrons, $r = |\mathbf{r}|$, e is the charge of the electron, and a is a constant with dimensions of length.

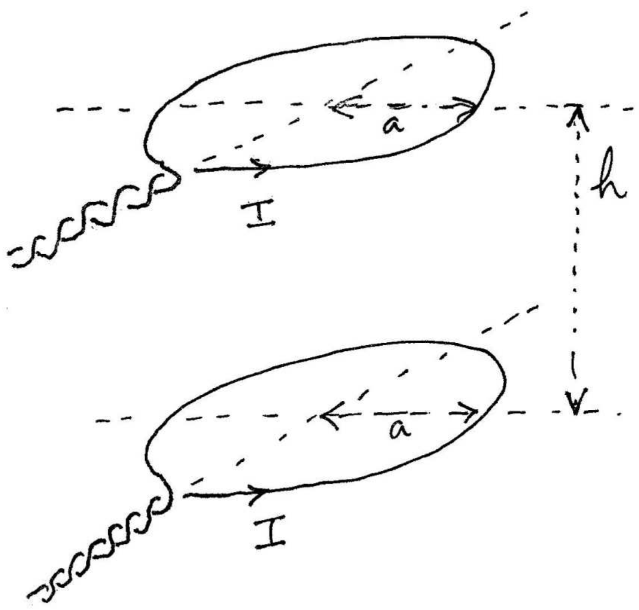
Find the Coulomb interaction energy between the two electrons. Simplify your final answer as much as possible. You may not merely quote the answer from memory or your cheat sheet.

Note: You may solve this problem in any way you like. One of many possible ways is as follows:
Step 1. Find the electrostatic potential of an infinitesimally thin spherical shell of charge.
Step 2. Find the electrostatic potential $\phi(\mathbf{r})$ due to the charge distribution $\rho_1(\mathbf{r})$ by superposing thin shells.
Step 3. Find the energy of the distribution $\rho_2(\mathbf{r})$ in the potential $\phi(\mathbf{r})$.

2. Consider the two circular current loops shown below. The loops are parallel, coaxial, of equal radius a, and separated by a distance h along their common axis. Both loops carry a current I in the same direction. The wires may be taken to have zero thickness.

(a) Find the force (magnitude and direction) between the two loops when $h << a$.
Hint: You only need the magnetic field due to a wire at points very close to that wire. For such points, the wire may be treated as infinitely long and straight. You must show how to find the field due to an infinite straight wire; it is not enough to quote it from memory.

(b) Find the force (magnitude and direction) between the two loops when $h >> a$.
Hint: Recall the general form of the far field of an arbitrary current loop, and the force on a small loop in an inhomogeneous external field.



3.(a) An electromagnetic plane wave of frequency ω is travelling in vacuum in the z direction, and reflected from a mirror located in the xy plane. Both the mirror and the wave may be taken to be of infinite extent in the x and y directions. The electric field in the incident wave is given by

$$E_{x,\text{inc}}(\mathbf{r}, t) = E_0 \sin(kz - \omega t - \alpha), \quad E_{y,\text{inc}} = E_{z,\text{inc}} = 0,$$

where α is an unknown phase, and $k = \omega/c$. Given that the *total* electric field vanishes at the mirror, find
(i) the electric field in the reflected wave,
(ii) the total electric field,
(iii) the magnetic field in the incident and reflected waves, and the total magnetic field.

(b) Repeat part (a) when the mirror is moving in the z direction at a uniform velocity u.
Hint: The boundary condition is the same as before, but it is now imposed at $z = ut$. Hence, the condition is $\mathbf{E}(x, y, ut, t) = 0$.

(c) One can think of the process of reflection as follows: The electromagnetic waves are absorbed and re-emitted by the mirror. In this way, use your answer for part (b) to derive the longitudinal Doppler shift for light. Answer, in particular, the following question: A light source is moving away from (or toward) an observer at a velocity u. The light frequency is ω in the source frame. What is the frequency measured by the observer?
Note: It is not enough to quote the Doppler shift from memory; you must derive it in the way asked for.

4. In the long-wavelength multipole expansion, the leading two terms for the far-zone electric field of a radiating system are given by

$$\mathbf{E}(\mathbf{R},t) = \begin{cases} \dfrac{1}{c^2 R}\Big[(\ddot{\mathbf{d}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}} - \ddot{\mathbf{m}} \times \hat{\mathbf{n}} + \cdots\Big]_{\text{ret}}, & \text{(Gaussian)}. \\ \dfrac{1}{4\pi\epsilon_0 c^2 R}\left[(\ddot{\mathbf{d}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}} - \dfrac{1}{c}\ddot{\mathbf{m}} \times \hat{\mathbf{n}} + \cdots\right]_{\text{ret}}, & \text{(SI)}. \end{cases}$$

Here, $\mathbf{d}$ and $\mathbf{m}$ are the electric and magnetic dipole moments of the radiating charges and currents, the dots indicate time derivatives, $\hat{\mathbf{n}} = \mathbf{R}/|\mathbf{R}|$, and the subscript 'ret' means that the quantity in square brackets is to be evaluated at the retarded time.

(a) Give the far-zone magnetic field in the same approximation. Simplify your answer as much as possible.

(b) Now consider a system for which

$$\mathbf{d} = d_0 \cos(\omega t)\, \hat{\mathbf{z}}, \quad \mathbf{m} = m_0 \cos(\omega t)\, \hat{\mathbf{z}}.$$

Find the time-averaged angular power distribution, i.e., find the time-averaged power radiated in a small solid angle $\Delta\Omega$ around the direction with spherical polar coordinates (θ, φ). (Recall that θ is measured from the z -axis.)

(c) For the system of part (b), find the total time-averaged power radiated over all directions.