Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Wednesday, June 8, 2011, 9 am - 1 pm

Electricity and Magnetism

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

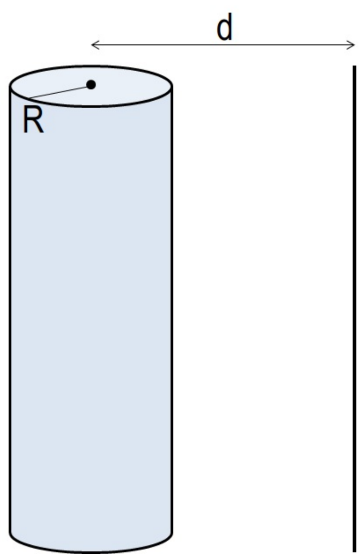
Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

1. An infinite straight wire with charge density λ is placed at a distance d from the symmetry axis of an infinite cylindrical conductor (radius $R < d$). The system, taken as a whole, is charge neutral.

   (a) Employ the method of images to solve for the electric potential and electric field. Verify that your solutions respect the spatial symmetries of the system.

   (b) Sketch equipotentials and field lines in a plane perpendicular to the wire.

   (c) Determine the charge density on the surface of the conductor.



2. The phase of a wave, $\phi = \omega t - \vec{k} \cdot \vec{x}$, is a Lorentz invariant.

   (a) Show that $k_\mu = (\omega/c, \vec{k})$ is a 4-vector.

   (b) How does k_μ transform for a boost given by $\vec{\beta}$?

   (c) Derive the formula for the relativistic Doppler shift relating ω' to ω in terms of γ, β, and $\cos\theta = \hat{\beta} \cdot \hat{k}$.

   (d) In the special case $\theta = \pi/2$ and $\gamma \gg 1$, show that $|\theta'\omega'/\omega| \approx 1$. This shows that as ω' becomes large relative to ω, θ' becomes small.

   (e) Consider two boosts with the same magnitude but opposite directions; $\pm\vec{\beta} \parallel \vec{k}$. This gives two different Doppler-shifted frequencies, $\omega_\pm$. Show that

$$\omega'_{\text{ave}} = \frac{1}{2}(\omega'_+ + \omega'_-) \neq \omega.$$

   What is $(\omega'_{\text{ave}} - \omega)/\omega$ to lowest order in $\beta \ll 1$?

3. Consider an induced molecular dipole

$$\vec{p} = -e\vec{x} = \frac{e^2}{m} \left(\omega_0^2 - \omega^2 - i\omega\gamma\right)^{-1} \vec{E} \tag{1}$$

where $\vec{E}$ is an electromagnetic wave, and the constants ω_0^2 and γ are characteristic of the molecule. In a particular sample of this molecule, there are N molecules per unit volume with Z electrons per molecule. The charge and mass of the electron are e and m.

(a) In the high-frequency limit, we can write

$$\frac{\epsilon(\omega)}{\epsilon_0} \approx 1 - \frac{\omega_p^2}{\omega^2}.$$

Derive an expression for ω_p^2 in terms of electron and molecular quantities.

(b) Derive a dispersion relation consistent with Eq. (1) and prove that $v_{\rm gr}\, v_{\rm ph} = c^2$ where $v_{\rm gr}$ is the group velocity and $v_{\rm ph}$ is the phase velocity. Use this to demonstrate that $v_{\rm gr} < c$ for $\omega > \omega_p$.

(c) For a tenuous plasma, Eq. (1) is valid even for $\omega < \omega_p$, in which case there is absorption.
What is the attenuation coefficient α as $\omega \to 0$?

4. A circular ring of current lies in the xy-plane and is centered on the z-axis. It has radius a and the steady current flows counter-clockwise (*i.e.*, in the direction of $\hat{\phi}$). Indeed, one can write a current density this way:

$$\vec{J} = -J_\phi \sin\phi' \, \hat{x} + J_\phi \cos\phi' \, \hat{y}$$

where

$$J_\phi = I \, \sin\theta' \, \delta(\cos\theta') \frac{\delta(r'-a)}{a}$$

and δ is the Dirac δ-function.

(a) Demonstrate that the magnetic field generated by the loop has the character of a dipole field, for $r \gg a$.
(*Caution:* a brute-force calculation is difficult and not needed. You may consider special points which render a calculation easier, such as a point on the axis of symmetry and in the plane of the loop.)

(b) Consider a long stack of these rings, all with the same radius and current, centered on the z-axis for all $z \leq 0$. If we take $a \to 0$ keeping Ia^2 constant, an unusual form for the magnetic field $\vec{B}$ is obtained. What is it?
(*Note:* You do not need to perform a very complicated calculation here, but be sure to justify your answer fully.)

Here are some formulae, for reference:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I \, d\vec{\ell} \times \vec{r}}{r^3}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \, d^3x' \qquad \nabla \times \mathbf{B}(\mathbf{x}) = \mu_0 \mathbf{J}(\mathbf{x})$$

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}) \qquad \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \, d^3x'$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_{\text{min}}^l}{r_{\text{max}}^{l+1}} \, Y_{lm}^*(\theta', \phi') \, Y_{lm}(\theta, \phi)$$

where $r_{\text{min}} = \min(\mathbf{x}, \mathbf{x}')$, similarly for r_{max}.