Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Wednesday, September 14, 2011, 9 am - 1 pm

Electricity and Magnetism

Solve 3 out of 4 problems

If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with “1 out of 2”, “2 out of 2” and so on.

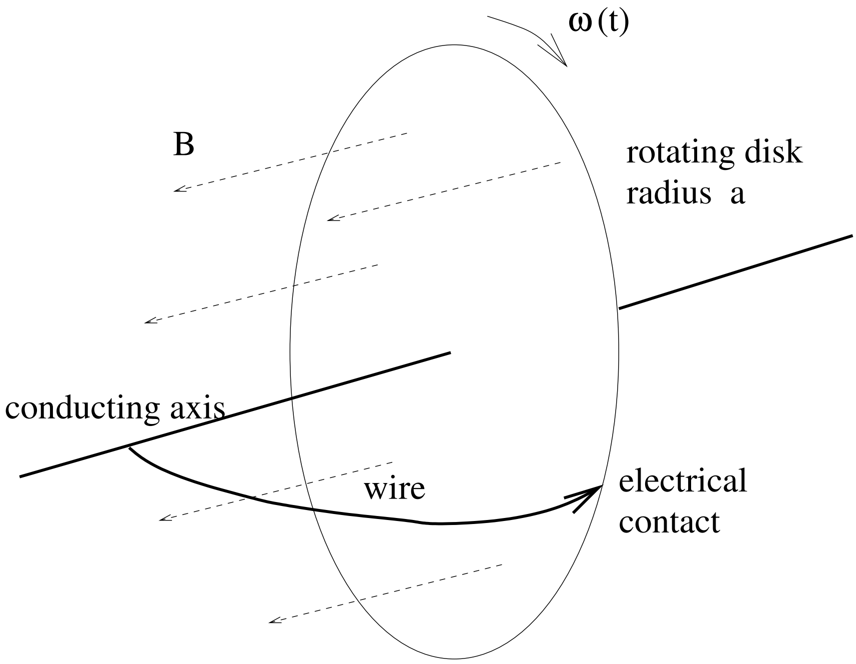
1. A metallic disk of radius a rotates at angular velocity $\omega(t)$; define $\omega_0 = \omega(0)$. It has mass M. A homogeneous magnetic field $\vec{B}$ intersects the disk as indicated by the figure. The direction of $\vec{B}$ is parallel to the axis of the disk. A conducting wire completes a circuit linking the center of the disk and its edge. This wire is stationary. The resistance of the complete circuit (disk-axis-wire) is R.

   The disk rotates without friction and with no driving force, for $t \geq 0$. Ignore eddy currents.

   (a) What is the direction and magnitude of the current flow?

   (b) What is the power dissipation at time $t \geq 0$?

   (c) What is the angular velocity as a function of time, $\omega(t)$, for $t \geq 0$? What is the characteristic time, τ, associated with $\omega(t)$?



2. A chief decay mode of the subatomic particle Λ is $\Lambda \longrightarrow p\pi$, where p is a proton and π is the pi-meson. They have masses M_Λ, m_p and m_π, respectively.

   Feel free to set $c \equiv 1$.

   (a) What is the momentum, p, of the proton and the pion in the rest frame of the Λ?

   (b) Find the energy E_p of the proton in the Λ rest frame, and take the limit $m_\pi \longrightarrow 0$.

   (c) Demonstrate that your answer in part (b) has the right values when you consider the limits $m_p \longrightarrow 0$ and $m_p \longrightarrow M_\Lambda$.

   (d) In the limit $m_\pi \longrightarrow 0$, show that

   $$\frac{E_p}{E_\pi} \approx \frac{M_\Lambda^2 + m_p^2}{M_\Lambda^2 - m_p^2}$$

   which shows that the proton gets most of the energy of the Λ.

   (e) Consider a large Lorentz boost characterized by $\gamma \gg 1$ and $\beta \approx 1$, and let E' denote the energy in the boosted reference frame. If the boost direction is parallel to the proton momentum vector, what is E'_p/E'_π? (Again take $m_\pi \rightarrow 0$.)

   (f) Repeat the calculation of part (e) when the boost is along the *pion* momentum vector. Show that the answer is approximately independent of β.

   (g) Consider again a Lorentz boost parallel to the proton momentum vector. For which boost factor γ will the pion be at rest in the boosted frame? Express your answer in terms of the masses M_Λ, m_p and m_π.

   If the decay has a low Q-value, so that $M_\Lambda^2 - m_p^2 - m_\pi^2 \rightarrow 0$, then γ has a rather simple form independent of m_π. What is that form, and what is the maximum value for γ?

   For part (g), do not neglect the pion mass (for obvious reasons).

3. A hollow rectangular wave guide is coaxial with the z-axis and extends for $0 < x < a$ and $0 < y < b$.

Consider TM (transverse magnetic) waves propagating in the $+z$-direction. The electric field $\vec{E}$ has a component in the z-direction which satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) \psi = 0$$

where $\psi = E_z$ and γ^2 is an eigenvalue depending on a and b. The boundary condition is $\partial\psi/\partial n = 0$ where $\partial/\partial n$ denotes the normal derivative.

(a) Find the solutions, *i.e.*, the eigenfunctions $\psi_{mn}(x, y)$ and the eigenvalues γ^2_{mn}.

(b) The dispersion relation for this waveguide is

$$k^2_{mn} = \mu_0\epsilon_0\,\omega^2 - \gamma^2_{mn}.$$

What is the cutoff frequency, below which the wave number is imaginary and $m = n = 1$? Express your answer in terms of a, b and the speed of light, c.

(c) What is the phase velocity, $v_{\rm ph}$, and group velocity, $v_{\rm gr}$, for general m and n? Assume ω is above the cutoff frequency.
Verify that $v_{\rm ph} \times v_{\rm ph} = c^2$.

(d) Compute the vector fields $\vec{E}$ and $\vec{H}$ from $E_z = \psi\, e^{i(kz-\omega t)}$, given that the transverse components of the fields are

$$\vec{E}_t = \frac{ik}{\gamma^2}\,\vec{\nabla}_t\,\psi \qquad \text{and} \qquad \vec{H}_t = \frac{\epsilon_0\omega}{k}\,\hat{z}\times\vec{E}_t$$

with $\gamma^2 = \mu_0\epsilon_0\,\omega^2 - k^2$, and again taking $m = n = 1$.

(e) Show that the wave has components *only* in the longitudinal direction (*i.e.* $+z$-direction) for any point along the axis of the tube.

4. Two particles enter a strong homogeneous field $\vec{B}$ with the same initial velocity and direction, $\vec{v} \perp \vec{B}$. The first particle has charge $Q_1 = q$ and mass $M_1 = m$. The second particle has charge $Q_2 = 2q$ and mass $M_2 = 2m$.

   (a) What are the radii, R_1 and R_2, of the circular trajectory followed by the two particles?

   (b) What are the angular frequencies of rotation in the magnetic field – the "cyclotron frequency" – for both particles?

   Due to the circular motion of the particles in the magnetic field, each particle radiates according to

$$P \approx \frac{\mu_0 \, \ddot{\mathrm{p}}^2}{6\pi c}$$

   where the dipole moment is $\vec{\mathrm{p}} = Q\,\vec{x}(t)$ and $\vec{x}(t)$ is the position of the particle. (Note: this is *not* the "intrinsic" dipole moment of the particle!)

   (c) What is the ratio P_2/P_1 of the radiated power for the second particle with respect to the first particle?

   (d) Find a condition such that the fractional energy loss per turn is much less than one.

   (e) What is the ratio of the change in the radii, $\Delta R_2/\Delta R_1$, after one rotation in the magnetic field?