Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Wednesday, June 6, 2012, 9 am - 1 pm

Electricity and Magnetism

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

1. Consider an empty metallic rectangular box with one corner at the origin and the opposite corner at (a, b, c). We are interested in the electric potential $\Phi(x, y, z)$ and related quantities inside the box. All sides are grounded except the one with $z = c$, which has $\Phi(x, y, c) = V$, a constant.

   (a) The standard solution for $\Phi(x, y, z)$ involves three separation constants, e.g. k_x, k_y and γ. What is γ in terms of k_x and k_y?

   (b) Find $\Phi(x, y, z)$ expressed as an infinite sum of terms (i.e., sum over integers n, m). Show that your answer satisfies all boundary conditions.

   (c) Compute $\vec{E}$ as a function of z for $x = a/2$, $y = b/2$.
   Confirm that your answer satisfies the symmetry of the boundary conditions.

   (d) Show explicitly that $\vec{E}$ is perpendicular to the surface of the plates at $z = 0$ and $z = c$, *for all* x *and* y.

   (e) Compute the surface charge density σ on the sides of the box at $z = 0$ and $z = c$.
   What is the maximum of σ at $z = c$?
   Show that σ vanishes at the boundary of the plate at $z = c$.

Recall

$$\int_0^a dx \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) = \frac{a}{2}\,\delta_{nn'}$$

and

$$\sum_{\text{all } n} \cos\left(\frac{n\pi x}{a}\right) = 0 \qquad \text{for} \qquad 0 < x < a$$

2. A spherical object of mass $m = 1\mu$g carries charge $q = 100$ nC. It is injected horizontally into a region of constant and homogeneous magnetic field $B = 100$ T; $\vec{B}$ points straight up. The initial velocity is $v_0 = 300$ m/s pointing in the $\hat{x}$ direction.

   In the questions below, give an algebraic expression or formula, and an approximate numerical answer (accurate to within a factor of two) using MKS units.

   (a) What is the initial radius of curvature, R_0, for the path taken by the object?

   (b) What is the initial cyclotron frequency, ω_0 and the period, T_0?

   (c) What is the radiated power P_0 in the initial revolution in the magnetic field? Compare the radiated energy, $P_0 \cdot T_0$, to the kinetic energy E_{kin}.

   (d) Although it starts out horizontal, the particle falls vertically downward through the force of gravity.

   Approximately how many revolutions will the object undergo before it falls $h = 2$ cm?

   How much energy E_{rad} will it be radiated during that time?

   Compare E_{rad} to the change in rotational energy $m(v_x^2 + v_y^2)/2$ due to the increase in v_z.

   (e) Show that the fractional energy loss per turn, $P{\cdot}T/E_{\text{kin}}$, is independent of velocity. What is its dependence on B, q and m?

Recall the Larmor formula:

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3}$$

and the following constants:

- $\epsilon_0 = 8.85 \times 10^{-12}$ C^2/N m^2 $\approx 10^{-11}$ C^2/N m^2
- $c = 3 \times 10^8$ m/s
- $g = 9.8$ m/s^2 ≈ 10 m/s^2
- $6\pi\epsilon_0 c^3 \approx 4.5 \times 10^{15}$ C^2/kg$\cdot$s

3. An artificial atom has one quantum level with frequency $\omega_0 > 0$ and damping factor $\gamma \ll \omega_0$. An electromagnetic wave with frequency ω propagates through a body of these artificial atoms. The phase velocity is $\omega/k = c/n$ and the group velocity is $d\omega/dk$.

The standard expression for ϵ/ϵ_0 is

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \left(\frac{Nq^2}{\epsilon_0 m}\right) \frac{1}{\omega_0^2 - \omega^2 - i\,\gamma\omega}.$$

(a) Approximate $\epsilon/\epsilon_0 \approx 1 - \omega_p^2/\omega^2$ for $\omega \gg \omega_0$. What is the constant ω_p^2 in terms of N, q, etc.?

(b) What are the real and imaginary parts of the index of refraction, $n = n_R + in_I = \sqrt{\epsilon_r}$, when $\omega = \omega_0$? (We are taking $\mu/\mu_0 = 1$.)
Hint: consider that if $z = \rho e^{i\phi}$ with $\rho \geq 0$, then $\sqrt{z} = \sqrt{\rho}\, e^{i\phi/2} \propto \cos\phi/2 + i\, \sin\phi/2$. Of course, $\cos^2 x/2 = (1 + \cos x)/2$ and $\sin^2 x/2 = (1 - \cos x)/2$.

(c) Show that if $\omega_p^2/\gamma\omega_0 \ll 1$, n_R is only slightly bigger than one, while n_I is much smaller than one. (We are still taking $\omega = \omega_0$.)
Is this behavior typical of a metal or of a dielectric, and why?

(d) What is the attenuation of the *amplitude* of the wave after it passes through a thickness L? Please give a general answer and then use your result from part (c). Notice the attenuation factor does not depend on ω_0 when $\omega = \omega_0$. How does the length L_0 for attenuation by a factor e^{-1} depend on the damping term γ?

(e) A general expression for the group velocity is

$$v_g = \frac{c}{n(\omega) + \omega \frac{dn}{d\omega}}.$$

Note that $dn/d\omega = (2/n) d\epsilon_r/d\omega$.
Compute $d\epsilon_r/d\omega$ for the example above, and evaluate for $\omega = \omega_0$.
Find a leading-order approximation (take $n = 1$) for the group velocity v_g. Is v_g less than or greater than c?

4. A parallel-plate capacitor is oriented parallel to the xz plane, as shown in the figure. The height is h, width is w and the separation is d. There is no dielectric between the plates and the fixed surface charge densities on the plates are $\pm\sigma$. Since $d \ll h, w$ we will ignore fringe fields.

   Let K be the rest frame of the capacitor, and K' be a frame moving with velocity $\vec{v} = v\,\hat{z}$ with respect to frame K.

   (a) What are the electric $\vec{E}$ and magnetic fields $\vec{B}$ in frame K?

   (b) What are the dimensions h', w' and d' in frame K'?

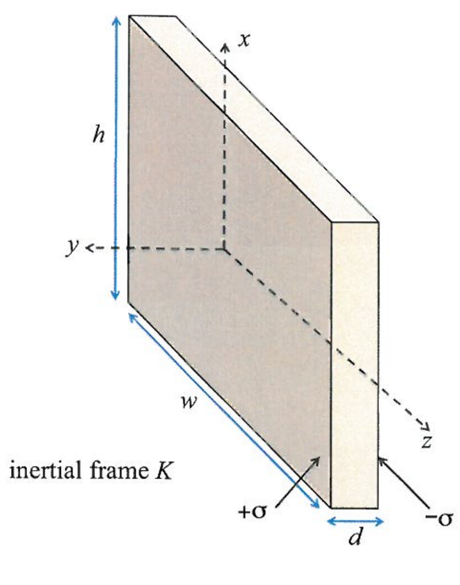
   (c) Calculate the electric field $\vec{E}'$ in the frame K' directly in terms the surface charge density σ.

   (d) Calculate the magnetic field $\vec{B}'$ in frame K' directly from surface currents.

   (e) Show explicitly that $E^2 - c^2B^2$ and $\vec{E} \cdot \vec{B}$ are the same in both frames.

   (f) A particle with charge q and mass m moves between the capacitor plates with initial velocity $\vec{u}' = u'\hat{z}$ in frame K'. What is the trajectory: y' as a function of z' (i.e., eliminate t')?

   (Hint: it may be easier to find the trajectory in a more convenient frame and transform the answer.)



Some notation: $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$; also, $1/c^2 = \epsilon_0\mu_0$.