Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Monday, September 14, 2015, 9 am - 1 pm

Electricity and Magnetism

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Note that you may use the Gaussian or SI systems of units for any problem, but please use the same system throughout the entirety of each problem.

Problem 1

A cylindrical shell of *imperfectly* conducting material with conductivity σ and height h fills the space between two concentric cylinder electrodes with radii a and b, $a < b$ (see figure). The inner electrode is held at voltage V_0 and the outer electrode is grounded. The surrounding material is perfectly insulating. The conductors have been held at this voltage for some time (*i.e.*, take the system to be in its steady state). Neglect any magnetic fields.

(a) Find $\nabla \cdot \vec{E}$ inside the conductor, *taking the system to be in its steady state*. In the steady-state, $\frac{\partial \rho}{\partial t} = 0$ everywhere in the conductor.

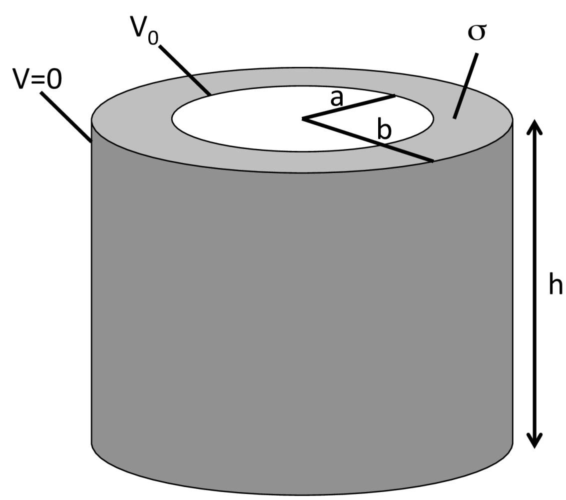
(b) Give boundary conditions on $\vec{E}$ (or on the the electric potential V) at all surfaces, infinitesimally inside the conductor. Include the top and bottom faces, taking the system to be in its steady state.
Hint: What happens if there are fringe fields inside the conductor?

(c) Find the electric potential V that satisfies the differential equation from part (a) and boundary conditions from part (b).

(d) Give the steady-state current density at all points in the conductor.

(e) Calculate the total current flowing between the two electrodes in the steady-state. What is the resistance between the electrodes?

(f) At no point have we assumed $h \gg b$. Explain why this assumption is not necessary for the calculation above (consider the hint in part (b)).

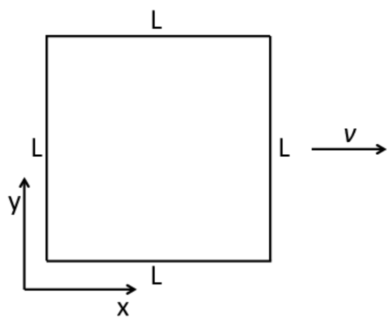


Problem 2

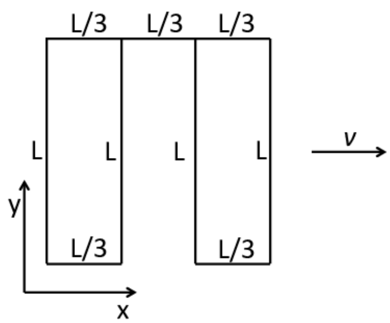
The wire networks in the figures below are being pulled at constant velocity $v \ll c$ in the $+x$ direction through the magnetic field $\vec{B} = B_0\left(\frac{x}{\lambda}\hat{\mathbf{z}} + \frac{z}{\lambda}\hat{\mathbf{x}}\right)$. The wire has a resistance per unit length of ρ. For each network:

(i) Find the induced current in each segment of wire in the network.

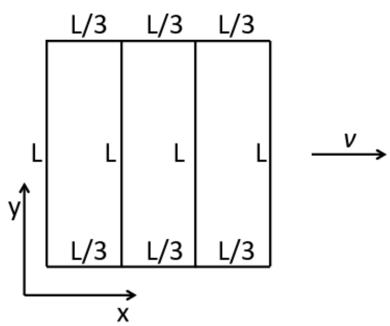
(ii) Find the net force on the wire network due to the magnetic field.



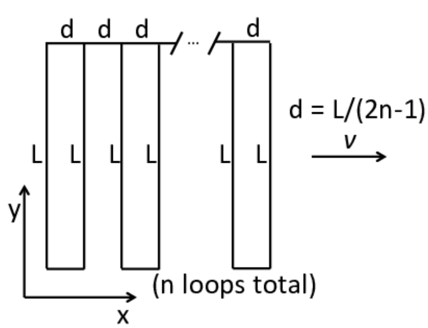
(a)



(b)



(c)



(d)

Problem 3

Consider a rectangular cavity with perfectly conducting walls and dimensions $a \times b \times c$, where $a > b > c$. The interior of the cavity is a perfect vacuum.

For simplicity, choose a coordinate system where one corner of the cavity is at the origin, with edges a, b, and c going along the $+x$, $+y$, and $+z$ axes, respectively.

(a) Give the boundary conditions on the electric and magnetic fields at the walls of the cavity.

(b) Find the lowest frequency standing wave in the cavity, giving the $\vec{E}$ and $\vec{B}$ fields everywhere in the cavity.

(c) Find the total energy stored in the cavity for the standing wave found in part (b).

(d) Find the surface currents $\vec{K}(\vec{r})$ in the walls of the cavity for the standing wave found in (b).

If the walls of the cavity are not quite perfectly conducting, there will be power loss due to the currents in the wall of the cavity, $P_{loss} = \int dA\, K^2 R_s$, where R_s is an effective surface resistance.

(e) Find the *average* rate of power loss $\langle P_{loss} \rangle$ in the cavity given the currents found in part (d).

(f) The quality factor of the cavity is given by $Q = \omega E_{stored}/\langle P_{loss} \rangle$. Find Q for the standing wave found in part (b).

Problem 4

Consider an antenna consisting of both an oscillating electric dipole, given by $\vec{p} = p_0 \cos \omega t \, \hat{\mathbf{z}}$, plus an oscillating magnetic dipole, $\vec{m} = m_0 \cos \omega t \, \hat{\mathbf{y}}$, where $m_0 = p_0 c$. The physical dimensions of the dipoles and the distance between the dipoles are all much smaller than c/ω.

(a) Find the electric and magnetic fields in the radiation zone.

(b) Find the angular distribution of the power radiated by the system. In what direction(s) is there no radiated power?

(c) Find the total, time-averaged momentum flux carried away by the electromagnetic radiation.

(d) What is the time-averaged net force on the pair of dipoles?

Possibly useful formulae:

$$\begin{aligned}
\vec{E}_p(\vec{r}, t) &= \frac{\mu_0}{4\pi r} \left[\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \ddot{\vec{p}} \right) \right] \\
\vec{B}_p(\vec{r}, t) &= -\frac{\mu_0}{4\pi r c} \left[\hat{\mathbf{r}} \times \ddot{\vec{p}} \right] \\
\vec{E}_m(\vec{r}, t) &= -\frac{\mu_0}{4\pi r c} \left[\hat{\mathbf{r}} \times \ddot{\vec{m}} \right] \\
\vec{B}_m(\vec{r}, t) &= -\frac{\mu_0}{4\pi r c^2} \left[\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \ddot{\vec{m}} \right) \right]
\end{aligned}$$

where the dipole moments are evaluated at time $t_0 = t - r/c$.