Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Monday, September 16, 2013, 9 am - 1 pm

Electricity and Magnetism

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Problem 1

A pair of infinite, concentric conducting cylinders with radii a and b ($a < b$) are held at electric potentials V_a and V_b, respectively.

(a) Find the electric field and electric potential in the region between the two cylinders, and determine the charge density on the surface of each cylinder.

(b) For a volume $\mathcal{V}$ bounded by a conducting surface S, show that

$$\int_{\mathcal{V}} d^3x\, \rho\widetilde{\Phi} + \int_S da\, \sigma\widetilde{\Phi} = \int_{\mathcal{V}} d^3x\, \widetilde{\rho}\Phi + \int_S da\, \widetilde{\sigma}\Phi,$$

where Φ is the potential due to the volume charge ρ and surface charge σ, and $\widetilde{\Phi}$ is the potential due to the volume and surface charges $\widetilde{\rho}$ and $\widetilde{\sigma}$. (This relation is known as Green's Reciprocation Theorem.)

Hint: Show that each side is equal to $\epsilon_0 \int_{\mathcal{V}} d^3x \left(\nabla\Phi \cdot \nabla\widetilde{\Phi}\right)$.

(c) Consider a point charge q sitting at radius r between the two cylinders. Use the relation from part (b) to find the total induced charge on each cylinder.

Hint: The induced charge is independent of V_a and V_b, so choose convenient values for these potentials.

Problem 2

Consider a magnetic dipole at the origin with dipole moment $m\hat{k}$ and velocity $-v\hat{k}$, with $0 < v \ll c$ ($\hat{k}$ is the unit vector in the $+z$ direction). The dipole is moving through a thin, cylindrical conducting tube of radius a. The cylinder's axis is the z-axis.

(a) Find the magnetic flux from the dipole through a circular cross-section of the tube at height z.

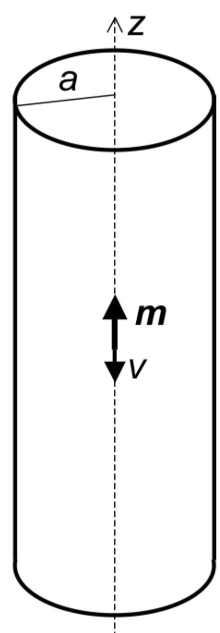
(b) Find the induced EMF in the tube from the moving dipole as a function of z.

(c) Think of the cylinder as a stack of rings, each with resistance R, with n rings per unit length. What is the induced current in a ring at height z? Let n become large with R/n fixed (continuum limit). What is the induced surface current in the tube as a function of z? Ignore any self-inductance of the tube.

(d) Write an expression for the force on the dipole due to the induced currents in the tube. Leave the answer as a 1-D integral. What is the direction of this force?

Possibly useful formula:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{r}}{r^3}$$



Problem 3

A plane wave traveling through a dielectric is incident on the boundary between the dielectric and vacuum. The dielectric has index of refraction $n > 1$ and magnetic permeability $\mu = \mu_0$. The incident wave is at an angle θ_i to the interface normal.

(a) For polarization *perpendicular* to the plane of incidence (Fig. 1), the relative amplitude of the reflected and incident waves is given by

$$\frac{E_r}{E_i} = \frac{n\cos\theta_i - \sqrt{1 - n^2\sin^2\theta_i}}{n\cos\theta_i + \sqrt{1 - n^2\sin^2\theta_i}}.$$

Use the boundary conditions on the $\vec{E}$ and $\vec{B}$ fields to find E_r/E_i for the case of polarization *parallel* to the plane of incidence (Fig. 2), in terms of n and θ_i. You may assume Snell's law.

(b) What is the condition for total internal reflection for each polarization case?

Let $\hat{\epsilon}_1$ denote polarization perpendicular to the plane of incidence (Fig. 1), and $\hat{\epsilon}_2$ denote polarization parallel to the plane of incidence (Fig. 2). Let $n = \sqrt{3}$ and consider an incident wave with $\theta_i = \pi/4$ and polarization $\frac{1}{\sqrt{2}}(\hat{\epsilon}_1 + \hat{\epsilon}_2)$.

(c) Describe in words (or with Stokes parameters, if you prefer) the polarization of the incident wave.

(d) The amplitude ratios in part (b) are correct even in the case of total internal reflection. Find E_r/E_i for polarizations $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ for the given n and θ_i. Express any complex values as an amplitude times a phase. What is the physical interpretation of the amplitude and phase of E_r/E_i?

(e) Find the polarization of the reflected wave given the incident polarization $\frac{1}{\sqrt{2}}(\hat{\epsilon}_1 + \hat{\epsilon}_2)$. Give your answer as a linear combination of $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ and describe the polarization in words (or with Stokes parameters).

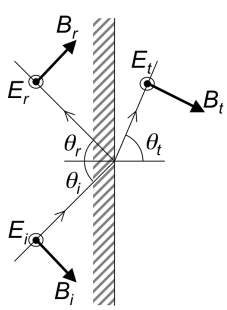


Figure 1

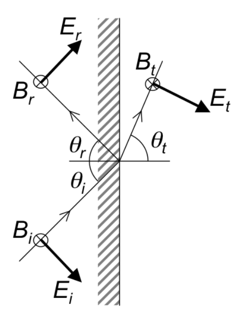


Figure 2

Problem 4

Consider a disc with radius R and charges $+q$ and $-q$ fixed at opposite points on the edge of the disc. The disc rotates without friction about its axis, which is fixed to the z-axis. For times $t < 0$ the disc rotates with constant angular velocity ω.

(a) What is the electric dipole moment of the system as a function of time for $t < 0$?

(b) Find the electric and magnetic fields in the radiation zone, to order $1/r$, from the oscillation in part (a).

Beginning at time $t = 0$ the disc is allowed spin freely, slowing down as it loses energy to electromagnetic radiation.

(c) Use your answer from (b) to find the rate at which energy is radiated from the disc at $t = 0$.

(d) As the disc slows, its angular momentum also decreases. Write an expression for the rate of angular momentum loss in terms of the power radiated.

(e) By conservation of angular momentum, the angular momentum loss in part (d) must be carried by the electromagnetic fields. Can this angular momentum loss be calculated from the fields in the answer to part (b)? Why or why not?