Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Tuesday, September 16, 2014, 9 am - 1 pm

Quantum Mechanics

Solve 3 out of 4 problems

If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with “1 out of 2”, “2 out of 2” and so on.

Problem 1

Consider two *distinguishable* spin-$\frac{1}{2}$ particles prepared in the pure spin state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(\,|{\uparrow\uparrow}\rangle + |{\downarrow\downarrow}\rangle\,\big)$$

where $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$ are the eigenstates of the S_z operator, and $|{\uparrow\uparrow}\rangle$ is shorthand for $|{\uparrow}\rangle_1 \otimes |{\uparrow}\rangle_2$.

(a) The spin of particle-1 is measured along an axis at angle θ_1 from the z-axis in the x-z plane, and the spin of particle-2 is simultaneously measured along an axis at angle θ_2 from the z-axis (also in the x-z plane). What are the possible outcomes of the two-particle measurement, and the probability of each?

(b) Suppose that instead of the pure state $|\psi\rangle$, the particles are in a mixed state, with equal probability of being either $|{\uparrow\uparrow}\rangle$ or $|{\downarrow\downarrow}\rangle$. What are the probabilities of the possible outcomes of the measurement from part (a) in this case?

(c) For what θ_1 and θ_2 can the scenarios in (a) and (b) can be distinguished? Give both a general condition and a specific example.

(d) Suppose you can only make a measurement on particle-1. Is there any choice of θ_1 that will distinguish (a) from (b)? Justify your answer.

Helpful fact:
For spin-$\frac{1}{2}$, the operator giving the spin-projection along the θ, ϕ axis has eigenstates:

$$|{\uparrow_{\theta,\phi}}\rangle = \cos\frac{\theta}{2}e^{-i\phi/2}\,|{\uparrow}\rangle + \sin\frac{\theta}{2}e^{i\phi/2}\,|{\downarrow}\rangle$$

$$|{\downarrow_{\theta,\phi}}\rangle = -\sin\frac{\theta}{2}e^{-i\phi/2}\,|{\uparrow}\rangle + \cos\frac{\theta}{2}e^{i\phi/2}\,|{\downarrow}\rangle$$

Problem 2

This problem considers a deuterium (^{2}H) atom in its ground state. The deuterium nucleus has spin $I = 1$.

(a) What is the degeneracy of the deuterium 1S state (n=1, l=0), including the electron and nuclear spins, before considering any spin-spin interactions?

(b) The interaction between the nuclear and electron spins adds a term

$$H_{S-I} = A\,\mathbf{I}\cdot\mathbf{S}$$

to the Hamiltonian, where $A > 0$, and $\mathbf{I}$ and $\mathbf{S}$ are the nuclear and electron spins, respectively. Find the 1S eigenstates of the new Hamiltonian, and give the energy-shifts and degeneracies of each.

(c) The atom now sits in a magnetic field B in the $+z$-direction, adding the term

$$H_B = \frac{Be}{m_e} S_z$$

to the Hamiltonian. Treating this term as a perturbation, find the energy shifts of the 1S states, to first order.

(d) For what magnetic field strength is your answer to (c) a good approximation?

If the magnetic field is strong enough, one must swap steps (b) and (c), making a basis using eigenstates of H_B and treating H_{S-I} as a perturbation.

(e) Without doing the full calculation, sketch an energy-level diagram for 1S states in the strong magnetic field case. Label states with appropriate quantum numbers.

Possibly useful formulae:

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|1,1;\frac{1}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|1,0;\frac{1}{2},\frac{1}{2}\right\rangle$$

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|1,-1;\frac{1}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|1,0;\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1,1;\frac{1}{2},-\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|1,0;\frac{1}{2},\frac{1}{2}\right\rangle$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = -\sqrt{\frac{2}{3}}\left|1,-1;\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|1,0;\frac{1}{2},-\frac{1}{2}\right\rangle$$

Problem 3

Consider a particle of mass m in a 1-dimensional square well, with the potential given by

$$V = \begin{cases} 0 & x < -a \\ -V_0 & -a \leq x \leq a \\ 0 & x > a \end{cases}$$

(a) Give the general solution to the time-independent Schrödinger equation for $x < -a$, $-a < x < a$, and $x > 0$, assuming a bound state ($E < 0$).

(b) Write the boundary conditions the wave function must satisfy at $x = \pm a$.

(c) How many bound states does the system have, in terms of m, V_0, and a? Support your answer with a graphical solution to the boundary conditions from part (b).

For parts (d) and (e), assume the particle begins in a bound energy eigenstate $\psi_n(x)$, where $n = 0$ is the ground state, $n = 1$ is the first excited state, etc.

(d) Suppose V_0 slowly (adiabatically) decreases from $\frac{3\pi^2\hbar^2}{a^2m}$ to $\frac{\pi^2\hbar^2}{a^2m}$. For which n will the particle remain bound, and for which will it escape the well?

(e) Suppose instead that the potential shift in (d) is instantaneous. What is the probability that the particle escapes the well? Write your answer in terms of the eigenfunctions of the system before and after the potential shift. (Do not solve for the explicit forms for the eigenfunctions.)

Problem 4

A heavy nucleus with spin $S = 2$ decays to a spin $S = 0$ nucleus via emission of two alpha-particles (also $S = 0$), each with orbital angular momentum $L = 1$. We want to find the probability distribution for the angle ω between the two alpha-particles.
(Note: *Linear* momentum is not a consideration in this problem — the nucleus will recoil to conserve linear momentum, but is sufficiently heavy that it can be considered to remain at rest.)

(a) Use addition of angular momentum to construct states with total angular momentum 2 from the two $L = 1$ particles (*i.e.*, what combinations of $Y_1^{m_1}(\theta_1, \varphi_1)Y_1^{m_2}(\theta_2, \varphi_2)$ give a basis for states with total angular momentum 2).

(b) Suppose the nucleus is initially in the state $S_z = 2\hbar$. What is the probability distribution $dP(\theta_1, \varphi_1; \theta_2, \varphi_2)/d\Omega_1 d\Omega_2$ for the directions of the emitted particles?

(c) Suppose now that the initial nucleus is unpolarized (that is, all S_z states have equal *probability*). What is the probability distribution for the directions of the emitted particles in this case? Leave your answer in terms of Y_ℓ^m's — do not simplify!

(d) Since the initial state in (c) is rotationally invariant, we are free to choose $\theta_1 = 0$. What is the probability distribution $dP(\theta_2, \phi_2)/d\Omega_2$ for the direction of the second alpha particle in this frame? Simplify your answer to this part, and show that it depends only on the angle between the two alpha particles.

Possibly useful formulae:

$$\begin{aligned}
Y_0^0(\theta, \varphi) &= \sqrt{\tfrac{1}{4\pi}} & Y_2^{\pm 2}(\theta, \varphi) &= \sqrt{\tfrac{15}{32\pi}} \sin^2\theta e^{\pm 2i\varphi} \\
Y_1^{\pm 1}(\theta, \varphi) &= \mp\sqrt{\tfrac{3}{8\pi}} \sin\theta e^{\pm i\varphi} & Y_2^{\pm 1}(\theta, \varphi) &= \mp\sqrt{\tfrac{15}{8\pi}} \sin\theta\cos\theta e^{\pm i\varphi} \\
Y_1^0(\theta, \varphi) &= \sqrt{\tfrac{3}{4\pi}} \cos\theta & Y_2^0(\theta, \varphi) &= \sqrt{\tfrac{5}{16\pi}} (3\cos^2\theta - 1)
\end{aligned}$$

$$L_\pm |l, m\rangle = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\, |l, m \pm 1\rangle$$