Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Thursday, June 9, 2011, 9 am - 1 pm

Quantum Mechanics

Solve 3 out of 4 problems

If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with “1 out of 2”, “2 out of 2” and so on.

1. One way to learn about the properties of a localized potential $V(\vec{r})$ is to "throw" known probes at it and determine how they scatter. Assume that the probe can be described by a plane wave, initially moving along the positive z-direction (wave vector $\vec{k} = k_0\hat{z}$, kinetic energy $E = \hbar^2 k_0^2/2m$) and that the potential can be parameterized by

$$V(\vec{r}) = \begin{cases} V_0 & \text{for } r \leq a, \\ 0 & \text{for } r > a. \end{cases} \tag{1}$$

   (a) Determine the differential scattering cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$ at leading order, using the Born approximation.

   (b) Qualitatively describe the behavior of your answer to (a) as a function of the scattering angle θ in the "low-energy" and "high-energy" limits, defined, respectively, by $k_0 a \ll 1$ and $k_0 a \gg 1$.

   (c) There are three important dimensionful parameters in this problem: V_0 – the strength of the potential, a – the range of the potential, and $E \propto k_0^2$, the energy of the probe. Under what circumstances is the Born approximation valid? Express your answer as an inequality among a, k_0, V_0 in the "low-energy" limit, $k_0 a \ll 1$ (the reasoning is that if you satisfy the Born approximation at low energies, you are certainly safe at high energies).

   (d) Your answer in (a) reveals that by measuring the scattering cross-section one can measure some function of the potential parameters V_0 and a. You will also notice that, at leading order, one is unable to determine the *sign* of V_0, i.e., whether the potential is repulsive or attractive. Qualitatively explain why this is the case. Would the inclusion of the next-to-leading order contribution (using the Born approximation) allow one to establish whether the potential is attractive or repulsive?

2. Glauber states are coherent states of the one-dimensional harmonic oscillator. They are also referred to as minimum-uncertainty states. Mathematically, they are defined as eigenstates of the lowering (or annihilation) operator $\hat{a}$:[1]

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{2}$$

where α is the *complex* eigenvalue.

(a) Compute the uncertainty relation for the coherent states: $\sigma_x\sigma_p$. Here $\sigma_{\mathcal{O}}^2 = \langle(\hat{\mathcal{O}} - \langle\hat{\mathcal{O}}\rangle)^2\rangle$ is the root-mean-square deviation of the operator $\hat{\mathcal{O}}$ from its mean. Comment on your result.

(b) $|\alpha\rangle$ are not eigenstates of the Hamiltonian, except for the ground state $|0\rangle$. The energy eigenstates, of course, are usually labelled $|n\rangle$, $n = \{0, 1, 2, 3, \ldots\}$, where $E_n = (n+1/2)\hbar\omega$. Compute $\langle\hat{x}\rangle_n(t)$ and $\langle\hat{p}\rangle_n(t)$.

(c) Compute the time evolution of a coherent state, showing that, as time progresses, even though the state may change, a coherent state remains a coherent state: $\hat{a}|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle$, where $\alpha(t)$ is a function of time.

(d) Compute $\langle\hat{x}\rangle_\alpha(t) = \langle\alpha(t)|\hat{x}|\alpha(t)\rangle$. Your answer will depend on the initial condition, $\alpha(t = 0) = \alpha_0$.

(e) Compute $\langle\alpha|\alpha'\rangle$ where $|\alpha\rangle$ and $|\alpha'\rangle$ are, in general, different coherent states. In light of your result, comment on the following "theorem:" *Two eigenvectors of an observable corresponding to two different eigenvalues are orthogonal.*

[1] We remind you that

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\hat{x} + i\sqrt{\frac{1}{m\omega}}\hat{p}\right),$$

where m and ω are, respectively, the mass and angular frequency of the harmonic oscillator. Using $\hat{a}$ (and its hermitian conjugate, $\hat{a}^\dagger$), the Hamiltonian is given by

$$\hat{H} = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hbar\omega.$$

3. (a) A particle of mass m is constrained to move in one-dimension in the presence of $V(x) = -u_0\delta(x - x_0)$, where $u_0 > 0$ and $\delta(x - x_0)$ is the usual Dirac delta function. Show that this system has only one bound state (state with a negative energy). Compute the energy and sketch the wave-function. You may choose $x_0 = 0$ in your computations.

   (b) Imagine that the potential is replaced by $V(x) = -u_0\left(\delta(x - x_1) + \delta(x - x_2) + \delta(x - x_3)\right)$, where $u_0 > 0$ and $x_1 \neq x_2 \neq x_3$. How many bound states are there *at most* in this case? [hint: you don't need to compute the eigenenergies, just count the different eigenstates, using your physics intuition].

   (c) Imagine that the potential is replaced by

$$V(x) = -u_0 \sum_{n=-\infty}^{+\infty} \delta(x - na),$$

   where $u_0 > 0$ and a is a constant distance. In order to determine the allowed energies for this system it is convenient to use Bloch's theorem, which states that if the potential is periodic: $V(x) = V(x+a)$ for some constant a, then the solutions to Schrödinger equation satisfy

$$\psi(x + a) = e^{iKa}\psi(x),$$

   where K is a real parameter that does not depend on x. Hence, it suffices to solve the Schrödinger equation inside a "cell" of length a and, using Bloch's theorem, define the wave-function in all space.

   Solve Schrödinger's equation with the help of Bloch's theorem, and obtain a relation between the eigenenergies and K (and a). Using the fact that K is real, qualitatively describe the allowed *negative* energy values.

   (d) In the scenario spelled out in (c), what conditions must the parameters of the system obey so that an $E = 0$ state is present?

4. A system with total angular momentum one ($j = 1$) is described by the Hamiltonian

$$H_0 = \Omega \left(J_z + \alpha \frac{J_z^2}{\hbar} \right),$$

where Ω, α are constants and J_z is the z-component of the total angular momentum. Assume that Ω is positive.

(a) What are the eigenvalues and eigenvectors of H_0? Express the latter in terms of the eigenstates of J_z, $|+1\rangle$, $|0\rangle$, $|-1\rangle$.[2] For what value(s) of α are some of the energy levels degenerate?

The system has a magnetic moment $\vec{M} = \gamma \vec{J}$. In the presence of a constant magnetic field $\vec{B}$, the Hamiltonian of the system is modified to

$$H = H_0 - \omega_0 \vec{J} \cdot \hat{B},$$

where $\omega_0 = \gamma |\vec{B}|$ and $\hat{B}$ is the unit vector that describes the direction of the magnetic field. Parameterize $\hat{B}$ using the standard "spherical-coordinates" angles θ, ϕ. Throughout, you may assume that ω_0 is positive.

(b) Henceforth, assume $\alpha = 1$ and $\omega_0 \ll \Omega$. Compute, to first order in ω_0, the eigenenergies of the system as a function of θ and ϕ.

(c) If $\vec{B}$ points along the y-direction, what are, at leading order, the eigenstates of the Hamiltonian?

(d) At $t = 0$, the system is prepared in the $m = 0$ state, $|0\rangle$. If $\vec{B}$ points along the y-direction, use the results you got for (b) and (c) to compute the probability P_{00}, as a function of time, that a J_z measurement will yield zero. Sketch P_{00}.

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[2] In more detail,

$$J_z |m\rangle = m\hbar |m\rangle,$$

$m = -1, 0, +1$. These are also eigenstates of J^2,

$$J^2 |m\rangle = 2\hbar^2 |m\rangle, \forall m.$$

Finally, $|m\rangle$ are "raised" and "lowered" by the raising and lowering operators, J_+ and J_-,

$$J_\pm |m\rangle = \hbar \sqrt{2 - m\,(m \pm 1)} |m \pm 1\rangle.$$